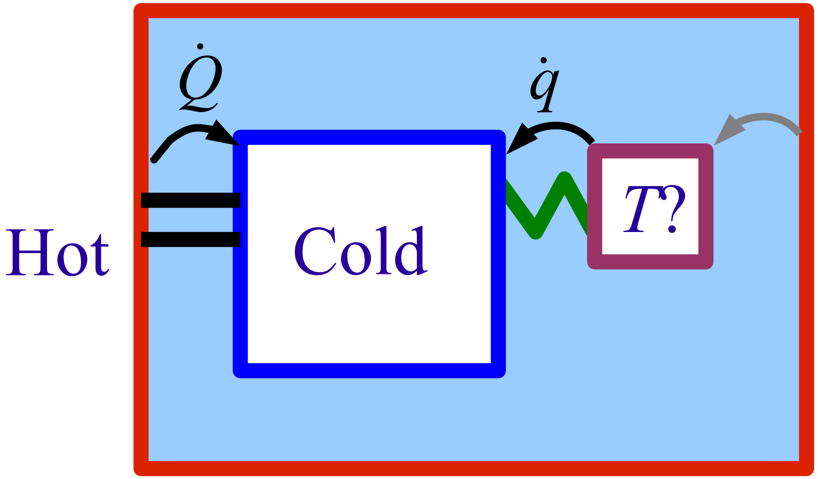
Heat Transfer

Topics and questions:

- Cooling time – "R in $\tau = RC$"
- Protection from external heating
- Thermometer reliability
- Existence of several temperatures: T_e, T_{ph}, T_n, ...
- Relationship between heat capacity and thermal conductivity
- Relationship between electrical and thermal conductivity



Sources:

P.V.E. McClintock, D.J. Meredith,
and J.K. Wigmore,
Low-Temperature Physics: an introduction
for scientists and engineers (1992)

F. Pobell,
Matter and Methods at Low Temperatures
(2007)

Mechanisms

- Thermal conduction
  - lattice vibrations (phonons)
  - conduction electrons (Fermi gas)
  - gas particles (“vacuum”)
- Thermal radiation (vacuum)
- Thermal convection (flow *i.e.* mass transfer)

Thermal conductivity κ relates temperature gradients ∇T to heat currents dQ/dt:

$$dQ/dt = -A\kappa\nabla T \qquad (\kappa = Cv^2\tau/3)$$

Simple sum rules: (carrier n / scattering effect i => τ_{ni})

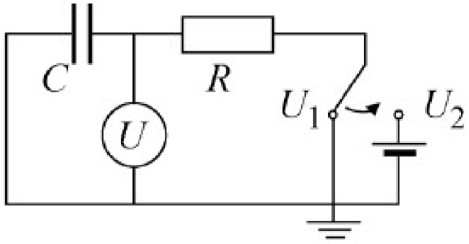
$1/\tau_n = 1/\tau_{n1} + 1/\tau_{n2} + \ldots$ (Mathiessen rule, sum over i)

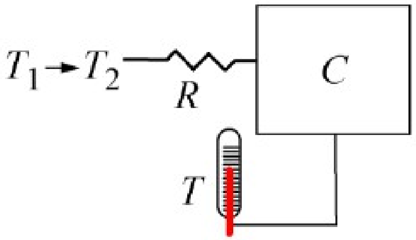
$\rho_n = \rho_{n1} + \rho_{n2} + \ldots$ $(\kappa_n = 1/\rho_n)$

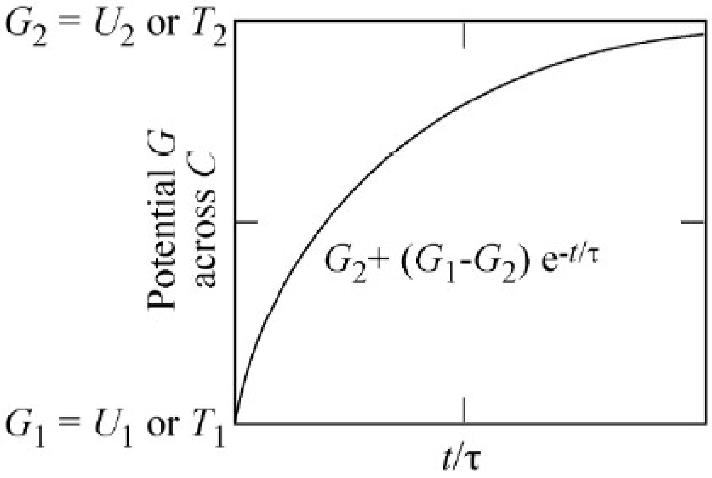
$\kappa = \kappa_1 + \kappa_2 + \ldots$ (sum over n)

An analogy

	electric	thermal
Potential:	U	T
Charge:	q	Q (heat)
Current:	$I = dq/dt$	dQ/dt
Capacitance:	$C = dq/dU$	$C = dQ/dT$
Conductivity:	σ	κ
Resistance:	$R = L/\sigma A$	$R = L/\kappa A$







Ohms law:

$\Delta U = RI$

$\Delta T = R(dQ/dt)$

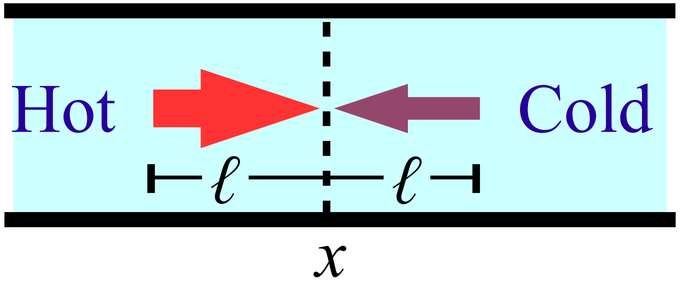
RC time constant:

$\tau = RC$ (both cases)

+ Kirchhoff's circuit laws can be applied!

Heat capacity and thermal conductivity

Kinetic theory relates the two (diffusion problem):



$$\frac{\dot{Q}}{A} = \frac{1}{2} nv\{E[T(x-\ell)] - E[T(x+\ell)]\}$$

Series expansion

$$\frac{\dot{Q}}{A} = \frac{1}{2} nv\left\{E[T(x)] + \frac{dE}{dT}\frac{dT}{dx}(-\ell) - E[T(x)] - \frac{dE}{dT}\frac{dT}{dx}(\ell)\right\}$$

$$= -nv\ell \frac{dE}{dT}\frac{dT}{dx} = -nv^2\tau \frac{dE}{dT}\frac{dT}{dx}$$

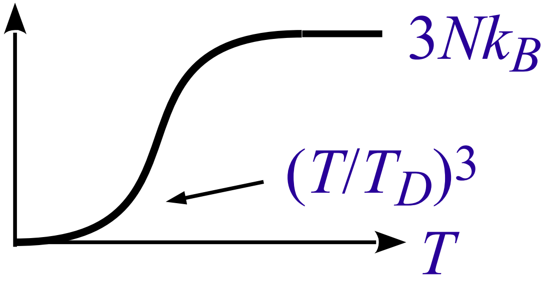
In 3D: $\langle v_x^2 \rangle = \langle v_y^2 \rangle = \langle v_z^2 \rangle = \frac{1}{3}\langle v^2 \rangle$ and $C = n\frac{dE}{dT}$

=> $\kappa = \frac{1}{3} C v^2 \tau = \frac{1}{3} C v \ell$

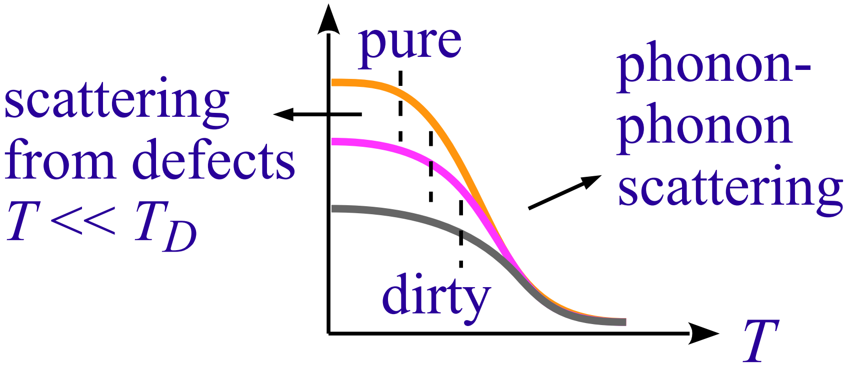
Phonons

Continuum approximation × C × ℓ

- v is constant (speed of sound) typically ~ 3 – 5 km/s
- specific heat: Debye theory

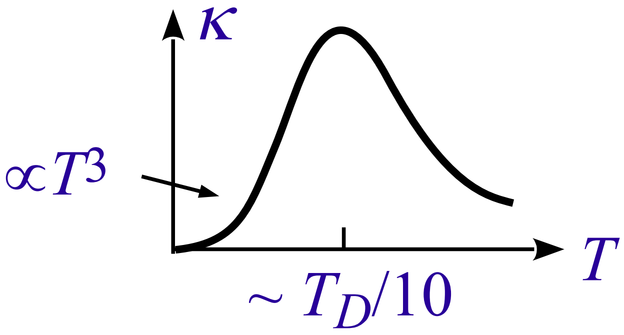


- mean free path:



=>

Thermal conductivity has a maximum at $T/T_\mathrm{D} \sim 0.1$



Examples

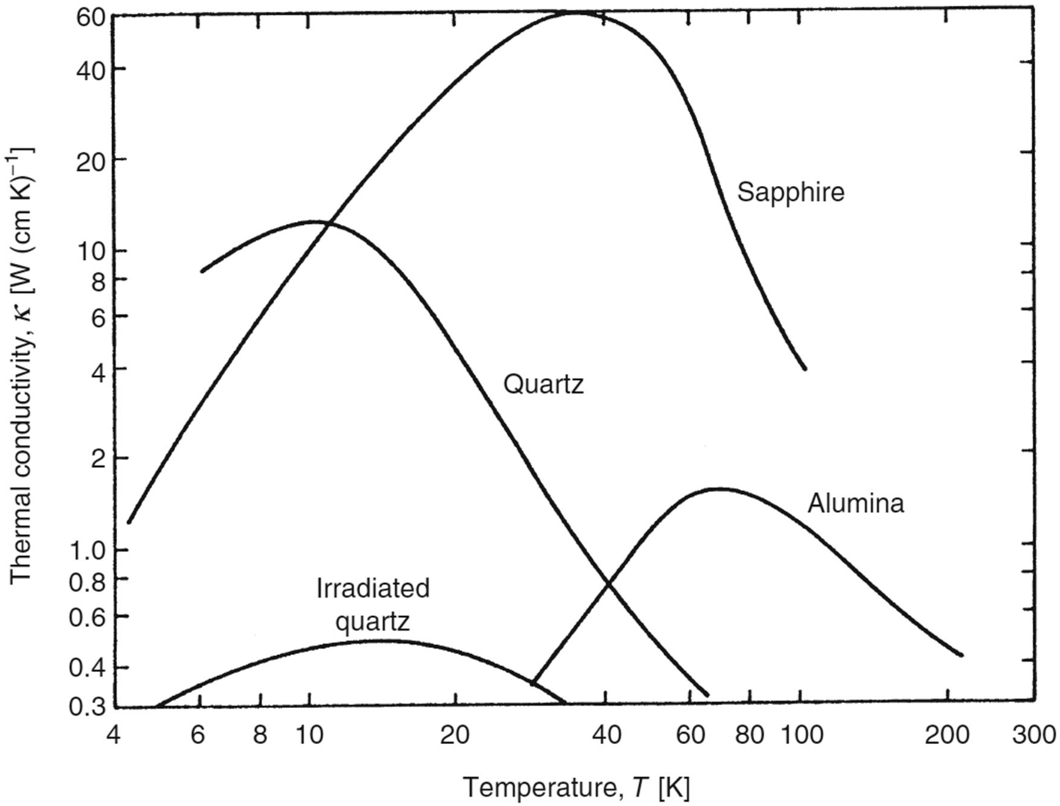


Fig. 3.19. Temperature dependence of the thermal conductivities of some dielectric solids

- Thermal conductivity of insulators (due to phonons) depends crucially on the crystal quality
- Big flawless single crystals may conduct as well as metals
- Amorphous solids have poor thermal conductivity

Conduction electrons

– $v = v_F \sim 100 \ldots 1000$ km/s

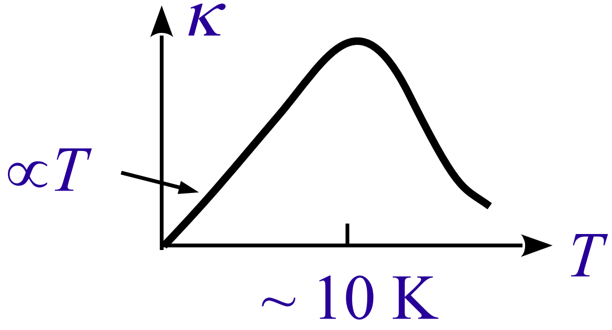
– specific heat $C \propto T$ (Sommerfeld theory)

– mean free path:

high T: electron-phonon scattering;
T increase => more phonons => κ decreases

low T: scattering from defects and impurities
ℓ is constant, $\kappa \propto C v \ell \propto T$

=> κ_{el} displays
a maximum
at around 10 K



Superconductors

Cooper pairs do not contribute to thermal conductivity ($C_{CP} = 0$)

Only normal electrons do and their number decreases exponentially far below T_c

$$n \propto \mathrm{e}^{-\Delta/k_BT}$$

$$\Rightarrow \quad \kappa \propto T\mathrm{e}^{-\Delta/k_BT} \sim T\mathrm{e}^{-2T_c/T}$$

Magnetic field can be used to destroy the super-conducting state

Used as heat switches!

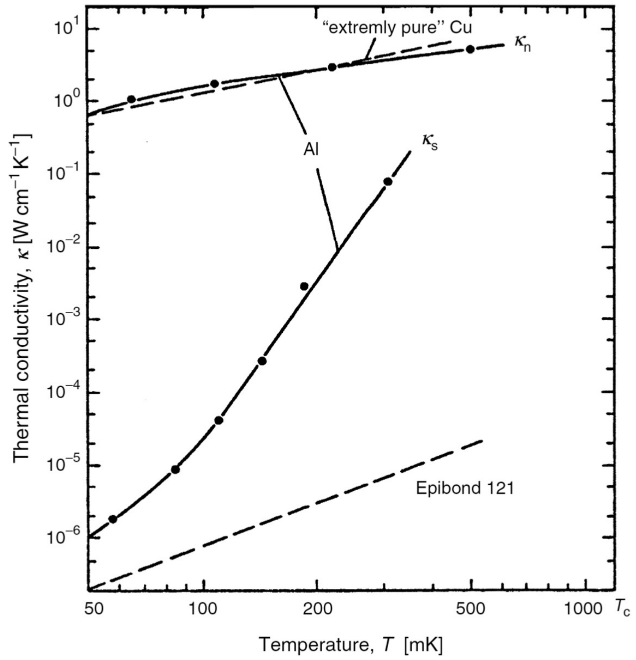


Fig. 4.1. Thermal conductivity κ of Al in the normal–conducting state (compared to κ of Cu) and in the superconducting state (compared to κ of the dielectric Epibond 121) at $T < 50$ mK [4.19]

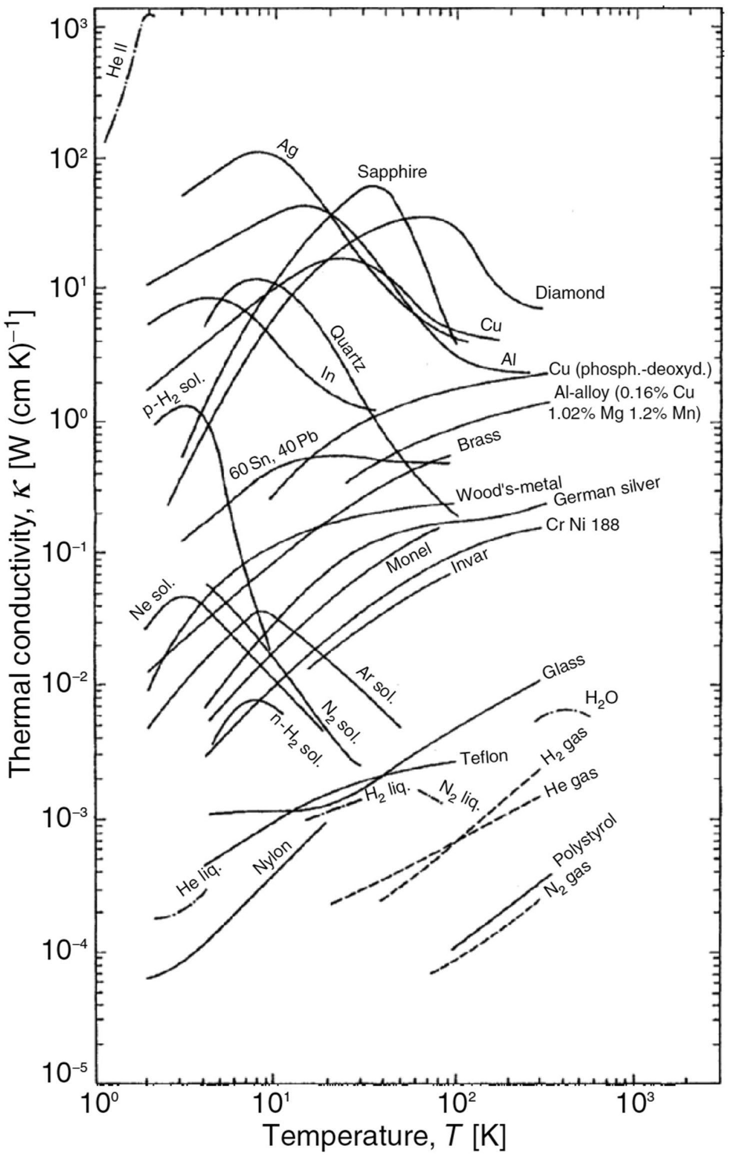


Fig. 3.20. Typical thermal conductivities κ of various materials at $T > 2\,\mathrm{K}$. Remember that κ depends on the purity and crystalline perfection of a material

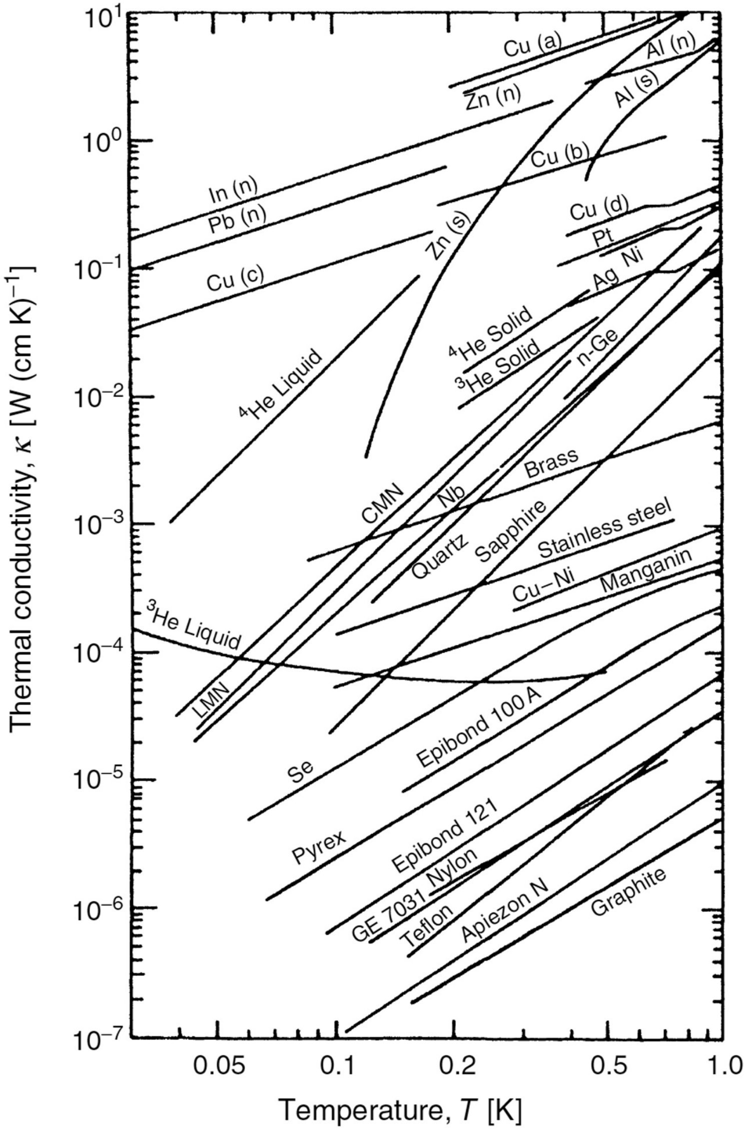
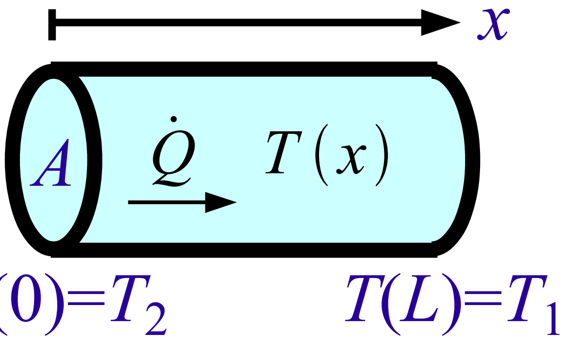


Fig. 3.21. Typical thermal conductivities κ of various materials at $T < 1\,K$.

Practical formulae for heat flow

$$\dot{Q} = -A\kappa \frac{dT}{dx}$$



$$\int_0^L \dot{Q}\,dx = -A\int_{T_2}^{T_1} \kappa\,dT$$

Metals: $\kappa = k_0 T$ for $T < 10$ K

$$\boxed{\dot{Q} = \frac{A k_0}{2L}(T_2^2 - T_1^2)}$$

Insulators: $\kappa = bT^3$ for $T < T_D/10$

$$\boxed{\dot{Q} = \frac{Ab}{4L}(T_2^4 - T_1^4)}$$

Note !!
do not use the linear approximation

$$\dot{Q} = \frac{A\kappa}{L}\Delta T$$

unless $\Delta T \ll T$

Higher temperatures

Use tabulated conductivity integrals, when the range of temperatures is large (*e.g.* from RT to 4K)

$$\bar{\kappa} = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \kappa(T)\, dT$$

$$\Rightarrow \dot{Q} = \frac{A}{L} (T_2 - T_1)\bar{\kappa}$$

$$T_1 = 4\,\mathrm{K}$$

T(K)	Copper (wire)	Stainless Steel	Glass	Teflon
6	800	0.63	0.211	0.113
10	3 320	2.93	0.681	0.44
20	14 000	16.3	2.0	1.64
50	50 800	135	8.46	7.16
77	68 600	317	17.5	13.0
100	80 200	528	29.2	18.7
140	97 600	939	54.2	28.7
200	122 000	1 660	103	44.2
300	162 000	3 060	199	70.2

in units W/m

Thermal (κ) and electric (σ) conductivities

Wiedemann-Franz law: $\kappa = \frac{1}{3} C v_F^2 \tau = \frac{\pi^2}{3} \frac{n k_B \tau}{m} T$

$$\sigma = \frac{n e^2 \tau}{m}$$

$$\Rightarrow \quad \frac{\kappa}{\sigma} = \frac{\pi^2}{3} \left(\frac{k_B}{e}\right)^2 T = L_0 T$$

with Lorenz number $L_0 = 24.5 \frac{\mathrm{nW\Omega}}{\mathrm{K}^2}$

Measure σ (easier) and use W-F to find κ

Wiedemann-Franz law

$$\boxed{\frac{\kappa}{\sigma} = L_0 T = T\,24.5\ \mathrm{nW\Omega/K^2}}$$

Condition for validity:

Both electric and thermal conductivities must be limited by the same scattering process

- works at room temperature (lots of phonons)
- works at very low temperatures (defect scattering)
- fails in between !

W-F failure at intermediate *T*

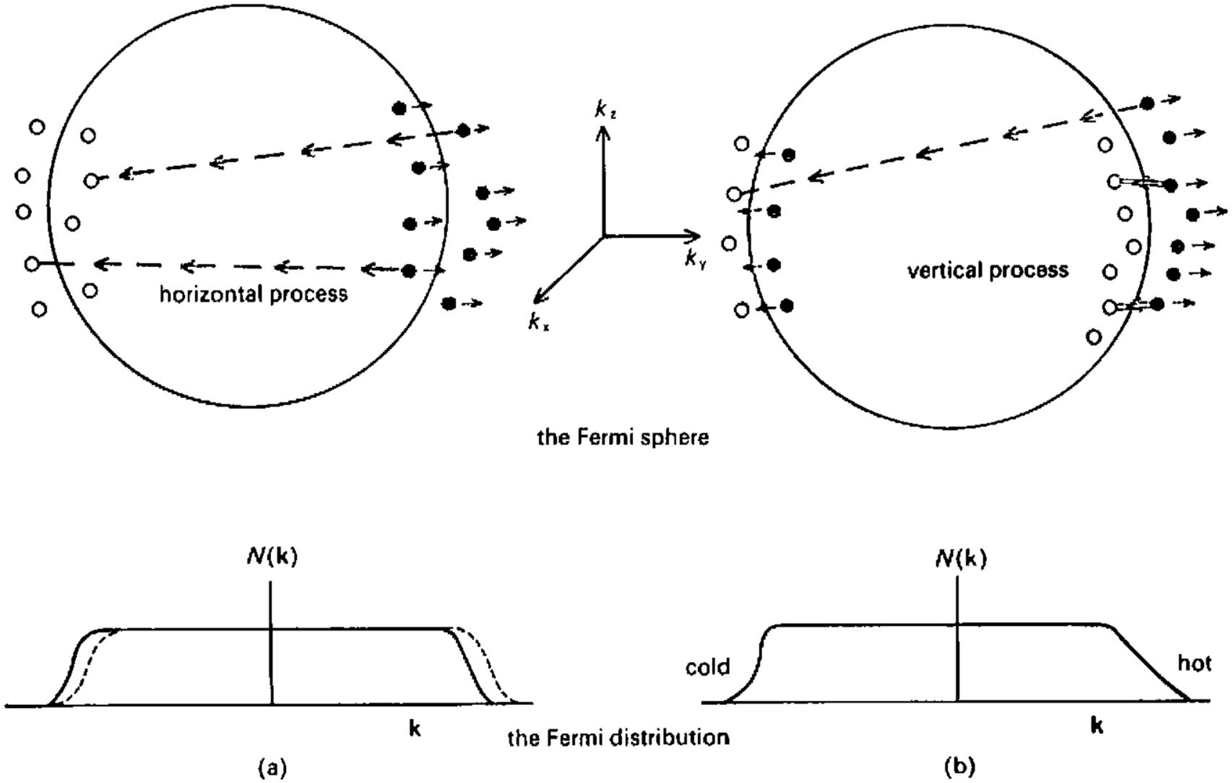


Figure 3.12 The disturbed Fermi distributions produced by the application of (*a*) an electric field to measure **J** and (*b*) a temperature gradient to determine **U**.
● excess of an electron relative to equilibrium distribution,
○ deficiency of an electron relative to equilibrium.
It is seen that electric field shifts the whole distribution in *k*-space, whereas temperature difference merely makes the distribution asymmetric.

Schematic
of the idea
(after McClintock et al.)

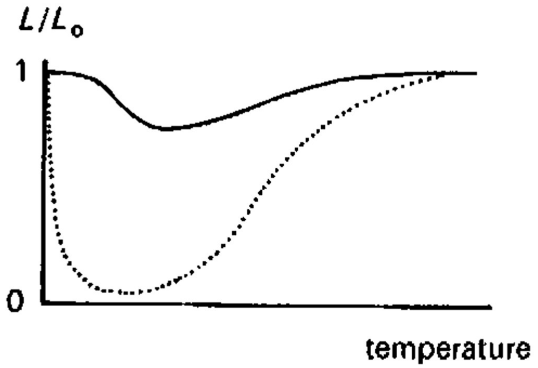
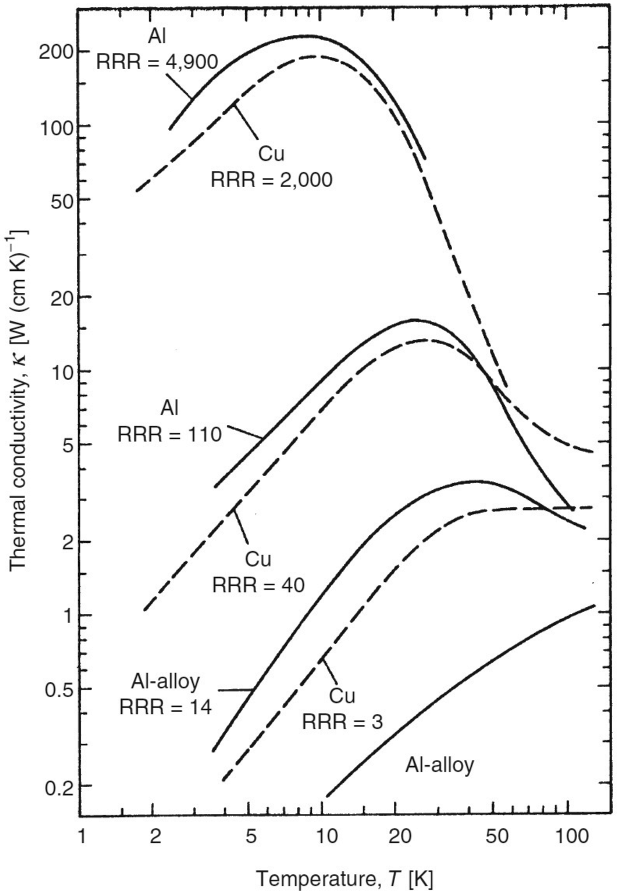


Figure 3.10 Dependence of the Lorenz number on temperature and sample purity.
——— impure sample;
.... highest purity crystal attainable.
(schematic)

Ways to improve κ

– Lattice imperfections can be reduced by simple annealing

– Impurities deteriorate thermal conductivity at low temperatures



Order of magnitude (for Cu) typically

- nonmagnetic: ~ 0.1 nΩcm/ppm
- magnetic (Cr, Fe, Co, ...)
  ~ 1 nΩcm/ppm

(remember $\sigma \rightarrow \kappa$ by W-F)

As a measure of purity one often uses the residual resistivity ratio: $RRR = R_{\mathrm{RT}}/R_{\mathrm{4K}}$

Oxygen annealing

In some cases (like Cu, Ag) magnetic impurities can be largely neutralized by annealing close to the melting temperature in low pressure oxygen atmosphere

- paramagnetic impurities oxidize to ferromagnetic compounds
- oxide molecules cluster into small crystals ($r \sim 0.1\ \mu m$)

=> less scatterers & smaller scattering cross section

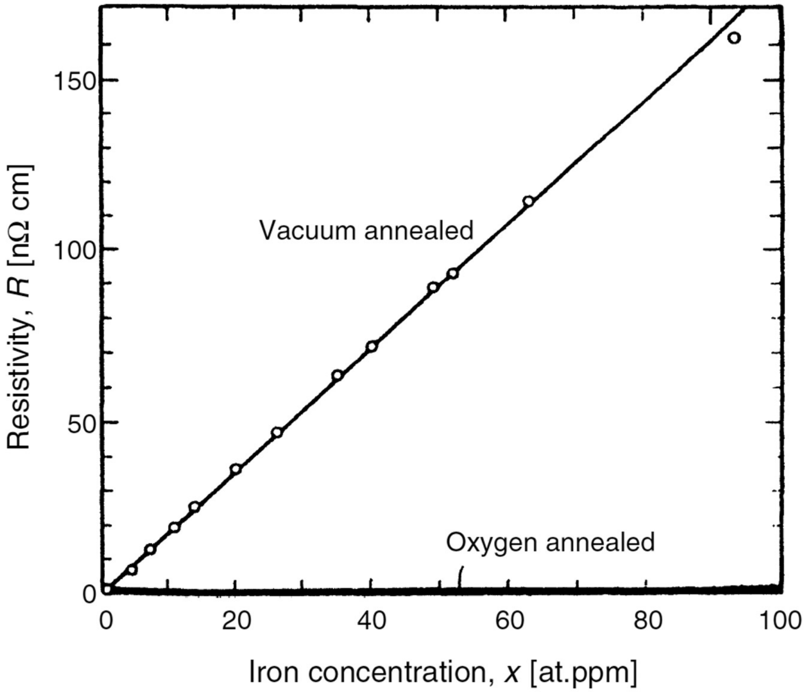


Fig. 3.26. Resistivity of Cu–Fe at 4 K, as a function of Fe impurity concentration, annealed in vacuum and in an O_2 atmosphere (92 h; about 10^{-6} bar air) [3.130]

Residual gas

1) dense gas; $\ell \ll D$ (size of the vacuum space)

$$\kappa = \frac{1}{3} C \bar{v} \ell$$

$$\ell \approx \frac{1}{\rho_N a_0^2}$$

molecules collide with each other

$\ell \propto 1/p$ (indep. of T)
$C \propto p$ (indep. of T)
$v \propto T^{1/2}$ (indep. of p)
=> κ is independent of p (but $\propto T^{1/2}$)

2) rarefied gas; $\ell > D$ (in fact effectively $\ell \rightarrow D$)

Knudsen limit or molecular transport regime
no collisions between the molecules
=> $\kappa \propto p$

Residual gas (cont.)

At 4 K or below only helium has significant vapor pressure

Vacuum space may be contaminated by

- poorly pumped heat exchange gas
- small leak
- helium diffusion through dielectric (soft) materials
  (rubber o-rings, epoxy feed throughs, glass-fiber neck, …)

Pressure as low as $p \sim 10^{-6}$ mbar is detrimental, since

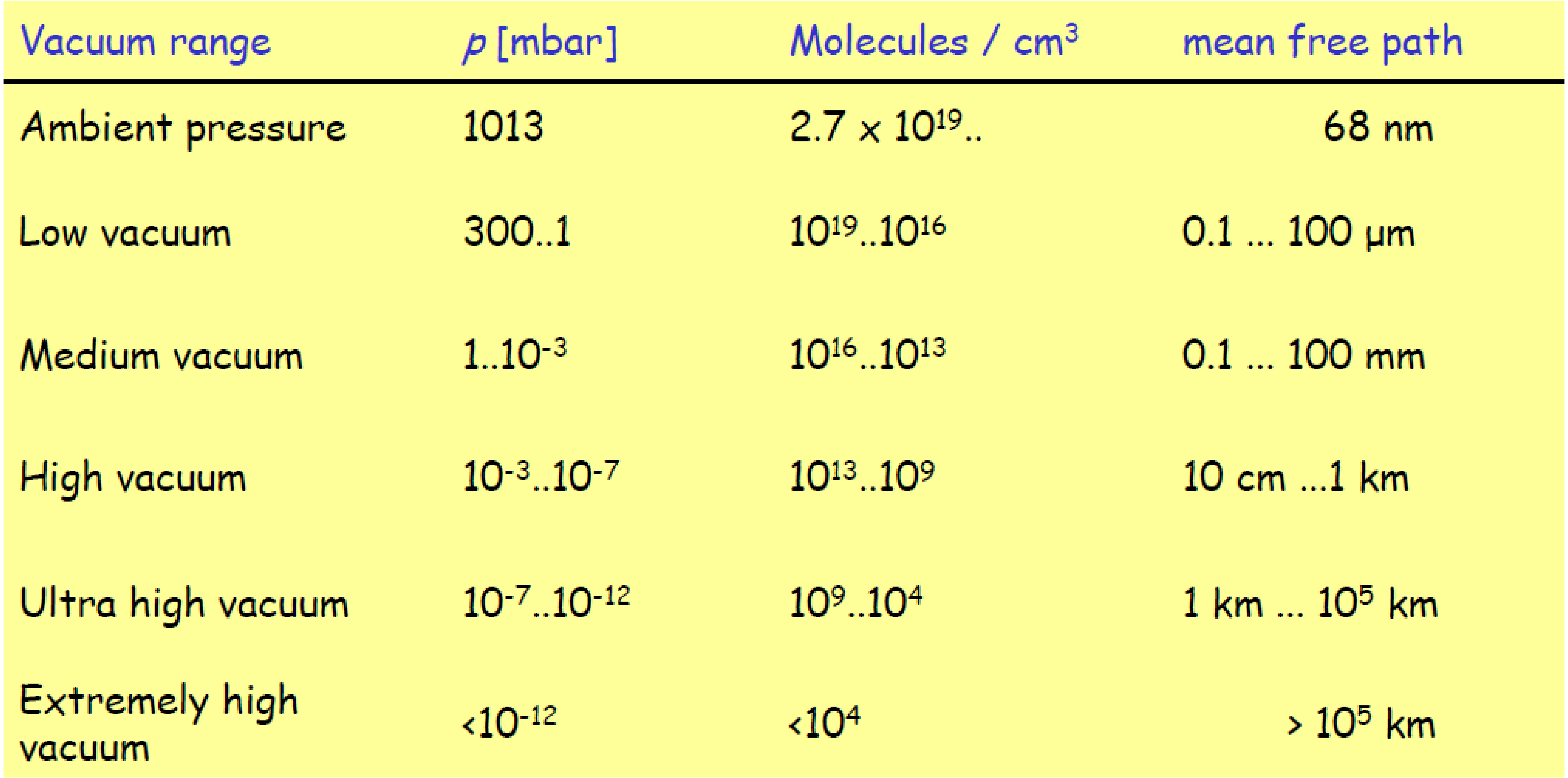
$$(dQ/dt)/\mathrm{W} \sim a\, A/\mathrm{cm}^2\, p/\mathrm{mbar}\, \Delta T/\mathrm{K}, \text{ with } a \sim 0.01 \text{ for He}$$

For example:

$$\left.\begin{array}{l} A = 600\ \mathrm{cm}^2 \\ T_1 = 4\ \mathrm{K} \\ T_2 \sim 0 \end{array}\right\} \Rightarrow dQ/dt \sim 25\ \mu\mathrm{W}$$

Vacuum can be improved by small activated charcoal pump in it

Mean free paths at reduced pressures



Vacuum range	*p* [mbar]	Molecules / cm^3	mean free path
Ambient pressure	1013	2.7×10^{19}..	68 nm
Low vacuum	300..1	$10^{19}..10^{16}$	0.1 ... 100 µm
Medium vacuum	$1..10^{-3}$	$10^{16}..10^{13}$	0.1 ... 100 mm
High vacuum	$10^{-3}..10^{-7}$	$10^{13}..10^{9}$	10 cm ...1 km
Ultra high vacuum	$10^{-7}..10^{-12}$	$10^{9}..10^{4}$	1 km ... 10^5 km
Extremely high vacuum	$<10^{-12}$	$<10^{4}$	$> 10^5$ km

Thermal radiation (black body radiation)

Stefan-Boltzmann law $\dot{Q} = A\sigma T^4$; $\sigma = 56.7 \frac{\text{nW}}{\text{m}^2\text{K}^4}$

Real surfaces are not “black”

emissivity ε (= absorptivity) is obtained from the index of reflectivity R : $\varepsilon = 1 - R$

$$\dot{Q} = A\sigma\left(T_1^4 - T_2^4\right)\frac{\epsilon_1\epsilon_2}{\epsilon_1+\epsilon_2-\epsilon_1\epsilon_2}$$

$$=> A\sigma\left(T_1^4 - T_2^4\right) \quad [\epsilon_1=\epsilon_2=1]$$

$$=> A\sigma\left(T_1^4 - T_2^4\right)\frac{\epsilon}{2} \quad [\epsilon_1\approx\epsilon_2=\epsilon\ll 1]$$

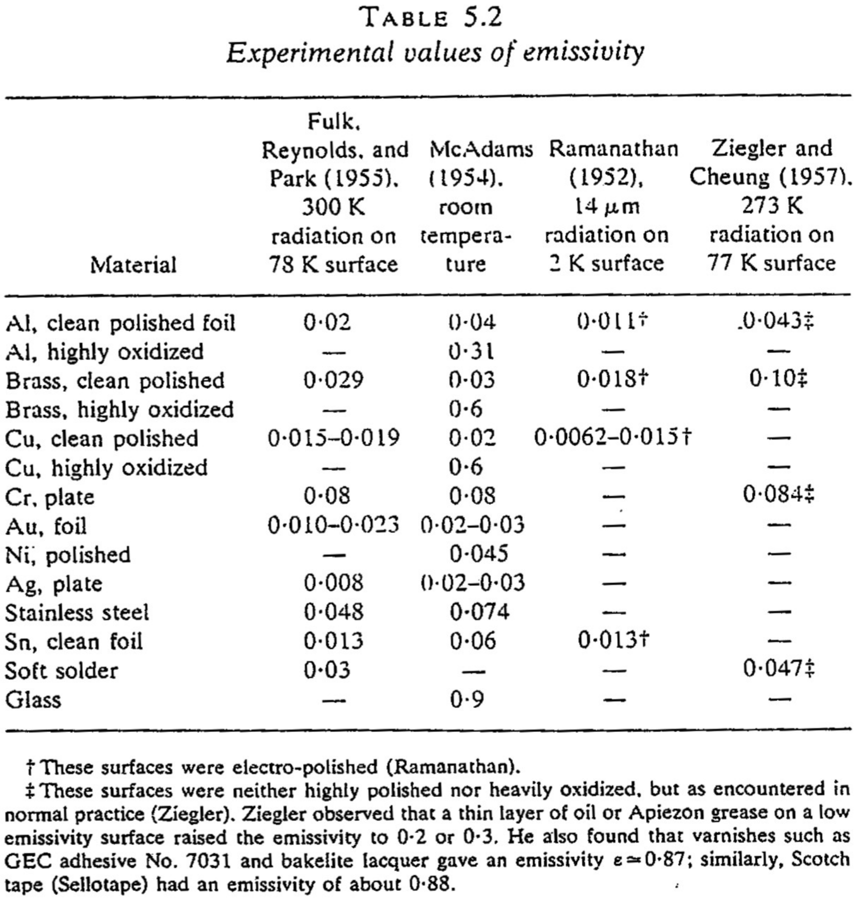


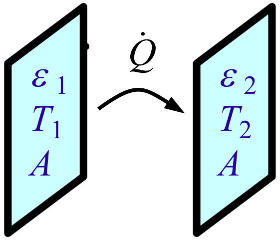
TABLE 5.2

Experimental values of emissivity

Material	Fulk, Reynolds, and Park (1955), 300 K radiation on 78 K surface	McAdams (1954), room temperature	Ramanathan (1952), 14 μm radiation on 2 K surface	Ziegler and Cheung (1957), 273 K radiation on 77 K surface
Al, clean polished foil	0·02	0·04	0·011†	0·043‡
Al, highly oxidized	—	0·31	—	—
Brass, clean polished	0·029	0·03	0·018†	0·10‡
Brass, highly oxidized	—	0·6	—	—
Cu, clean polished	0·015–0·019	0·02	0·0062–0·015†	—
Cu, highly oxidized	—	0·6	—	—
Cr, plate	0·08	0·08	—	0·084‡
Au, foil	0·010–0·023	0·02–0·03	—	—
Ni, polished	—	0·045	—	—
Ag, plate	0·008	0·02–0·03	—	—
Stainless steel	0·048	0·074	—	—
Sn, clean foil	0·013	0·06	0·013†	—
Soft solder	0·03	—	—	0·047‡
Glass	—	0·9	—	—

† These surfaces were electro-polished (Ramanathan).

‡ These surfaces were neither highly polished nor heavily oxidized, but as encountered in normal practice (Ziegler). Ziegler observed that a thin layer of oil or Apiezon grease on a low emissivity surface raised the emissivity to 0·2 or 0·3. He also found that varnishes such as GEC adhesive No. 7031 and bakelite lacquer gave an emissivity $\varepsilon \simeq 0{\cdot}87$; similarly, Scotch tape (Sellotape) had an emissivity of about 0·88.



Thermal radiation (cont.)

Operates even through perfect vacuum

Important above ~ 1 K (dies away as T^4)

To reduce its effect:

- Shiny surfaces; (high emissivity) silver coating →
- Superinsulation; ← aluminized mylar
- Multiple fixed radiation shields 77 K, 4 K, 1 K, 0.1 K

Dewar bottle

Helium fluids

^{4}He above $T_\lambda \sim 2$ K and ^{3}He above ~ 0.1 K behave more or less like classical gases with $\kappa \propto T^{1/2}$

^{4}He at ~ 1 ... 2 K has huge thermal conductivity because of normal-superfluid counterflow

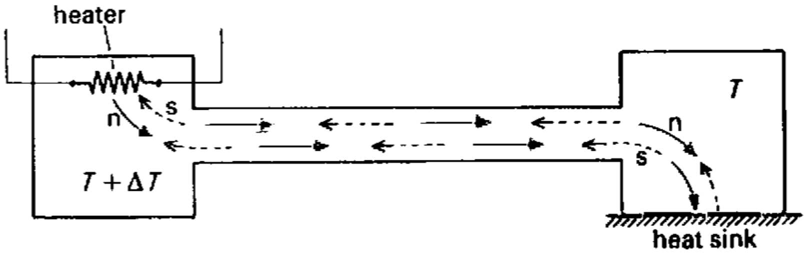
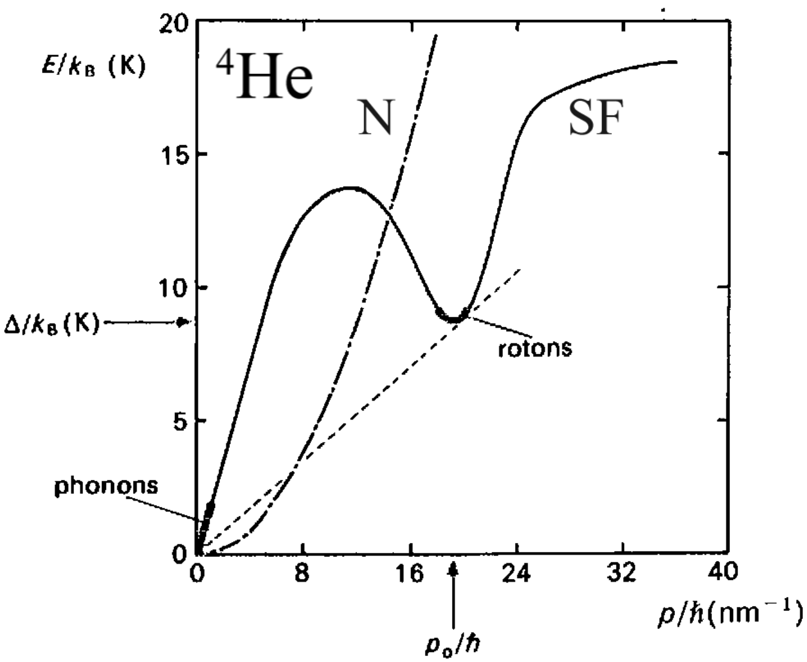


Figure 5.8 Sketch to indicate the peculiarly effective mode of thermal conduction in generated at the heater converts superfluid into normal fluid, which flows (full arrows) sink. Here, it gives up all of its entropy in becoming re-converted to superfluid, whi (dashed arrows) once more to the heater.

Below ~ 0.5 K ballistic phonons result in $\kappa \propto DT^3$



^{3}He

^{3}He and helium mixtures at low temperatures ($T << T_F$) are fermi systems with

$$\left.\begin{array}{l} \ell \propto 1/T^2 \\ v = v_F \\ C \propto T \end{array}\right\} \Rightarrow \;\; \kappa \propto 1/T$$

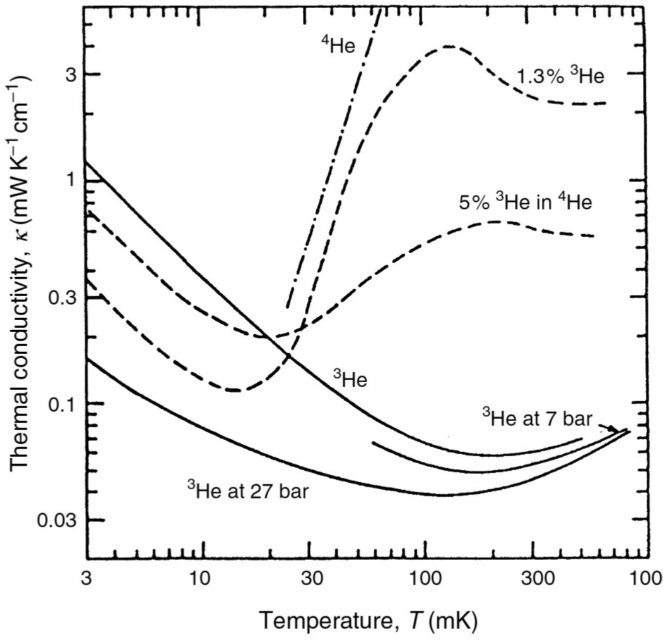


Fig. 2.18. Thermal conductivities of liquid ^{3}He, liquid ^{4}He, and dilute ^{3}He–^{4}He mixtures

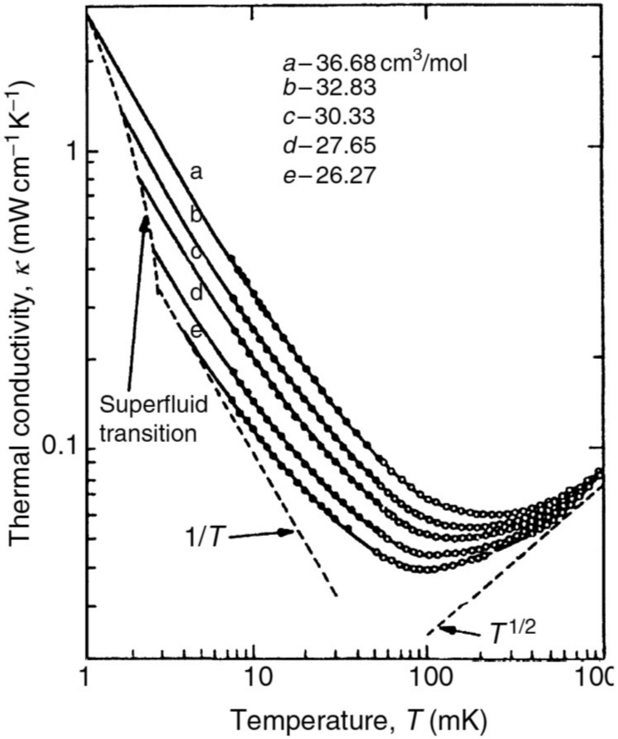


Fig. 2.17. Thermal conductivity of liquid ^{3}He at the given molar volumes At low temperatures, one finds the Fermi-liquid behavior $k \,\alpha\, T^{-1}$

Reference table for thermal conductivities

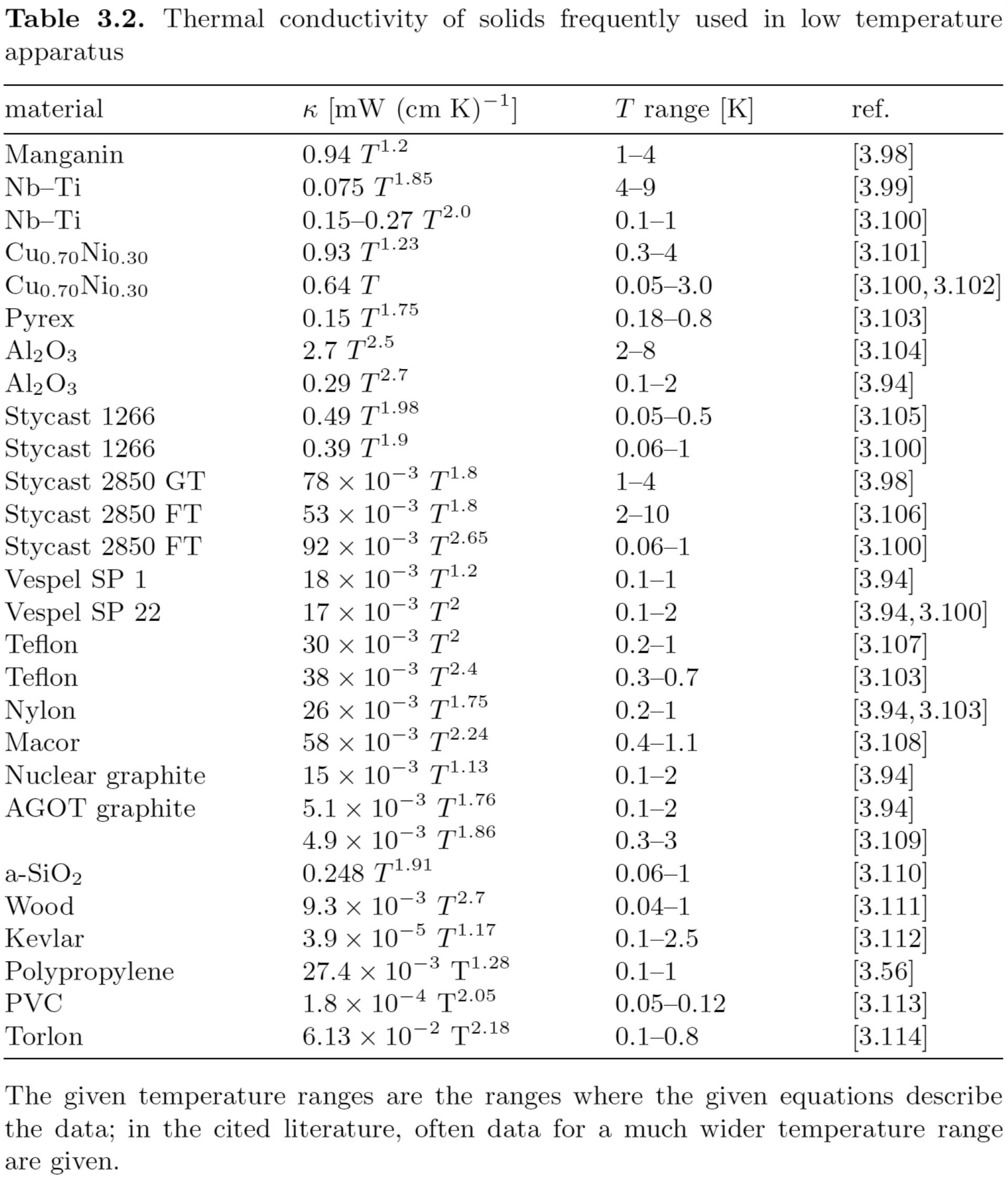


Table 3.2. Thermal conductivity of solids frequently used in low temperature apparatus

material	κ [mW (cm K)$^{-1}$]	T range [K]	ref.
Manganin	$0.94\ T^{1.2}$	1–4	[3.98]
Nb–Ti	$0.075\ T^{1.85}$	4–9	[3.99]
Nb–Ti	0.15–$0.27\ T^{2.0}$	0.1–1	[3.100]
$Cu_{0.70}Ni_{0.30}$	$0.93\ T^{1.23}$	0.3–4	[3.101]
$Cu_{0.70}Ni_{0.30}$	$0.64\ T$	0.05–3.0	[3.100, 3.102]
Pyrex	$0.15\ T^{1.75}$	0.18–0.8	[3.103]
Al_2O_3	$2.7\ T^{2.5}$	2–8	[3.104]
Al_2O_3	$0.29\ T^{2.7}$	0.1–2	[3.94]
Stycast 1266	$0.49\ T^{1.98}$	0.05–0.5	[3.105]
Stycast 1266	$0.39\ T^{1.9}$	0.06–1	[3.100]
Stycast 2850 GT	$78 \times 10^{-3}\ T^{1.8}$	1–4	[3.98]
Stycast 2850 FT	$53 \times 10^{-3}\ T^{1.8}$	2–10	[3.106]
Stycast 2850 FT	$92 \times 10^{-3}\ T^{2.65}$	0.06–1	[3.100]
Vespel SP 1	$18 \times 10^{-3}\ T^{1.2}$	0.1–1	[3.94]
Vespel SP 22	$17 \times 10^{-3}\ T^{2}$	0.1–2	[3.94, 3.100]
Teflon	$30 \times 10^{-3}\ T^{2}$	0.2–1	[3.107]
Teflon	$38 \times 10^{-3}\ T^{2.4}$	0.3–0.7	[3.103]
Nylon	$26 \times 10^{-3}\ T^{1.75}$	0.2–1	[3.94, 3.103]
Macor	$58 \times 10^{-3}\ T^{2.24}$	0.4–1.1	[3.108]
Nuclear graphite	$15 \times 10^{-3}\ T^{1.13}$	0.1–2	[3.94]
AGOT graphite	$5.1 \times 10^{-3}\ T^{1.76}$	0.1–2	[3.94]
	$4.9 \times 10^{-3}\ T^{1.86}$	0.3–3	[3.109]
a-SiO_2	$0.248\ T^{1.91}$	0.06–1	[3.110]
Wood	$9.3 \times 10^{-3}\ T^{2.7}$	0.04–1	[3.111]
Kevlar	$3.9 \times 10^{-5}\ T^{1.17}$	0.1–2.5	[3.112]
Polypropylene	$27.4 \times 10^{-3}\ \mathrm{T}^{1.28}$	0.1–1	[3.56]
PVC	$1.8 \times 10^{-4}\ \mathrm{T}^{2.05}$	0.05–0.12	[3.113]
Torlon	$6.13 \times 10^{-2}\ \mathrm{T}^{2.18}$	0.1–0.8	[3.114]

The given temperature ranges are the ranges where the given equations describe the data; in the cited literature, often data for a much wider temperature range are given.

Isolation and contact

- ISOLATION from the “hot” environment
  1) thermal conduction
    - support structures
    - measurement wiring
    - residual gas in the vacuum space
  2) thermal radiation
  3) power generated in the low temperature assembly
    - mechanical vibrations
    - electromagnetic interference
    - measurements themselves
    - etc …
- CONTACT between samples, thermometers and refrigerants
    - proper choice and preparation of the materials
    - joints and contact resistance
    - effects of thermal expansion

Ranges of temperature

Problems at high-*T* and low-*T* stages are different

T-range:	100 … 1 K	1 … 0.01 K	10 … 0.1 mK ...
Cooler:	He bath/pulse tube	dilution fridge	nuclear demag.
Typical size:	20 … 200 kg	5 … 50 kg	1 … 5 kg
Typical power:	W … mW	mW … µW	µW … nW … pW
Main source:	radiation/ conduction	conduction/ dissipation	dissipated power
CONTACT:	quite easy	some trouble	difficult
ISOLATION:	bit difficult	quite easy	quite easy

Heat switches

Needed for:

1) Initial cooldown from room temperature
   - – huge amount of heat must be removed
   - – low-T stages effective only below certain T_s

   Usual method: heat exchange gas in vacuum space
   - – N_2 or Ne to below 100 K, ^{3}He to 10 K
   - – pump vacuum space carefully!
   - – small activated charcoal pump can improve the vacuum

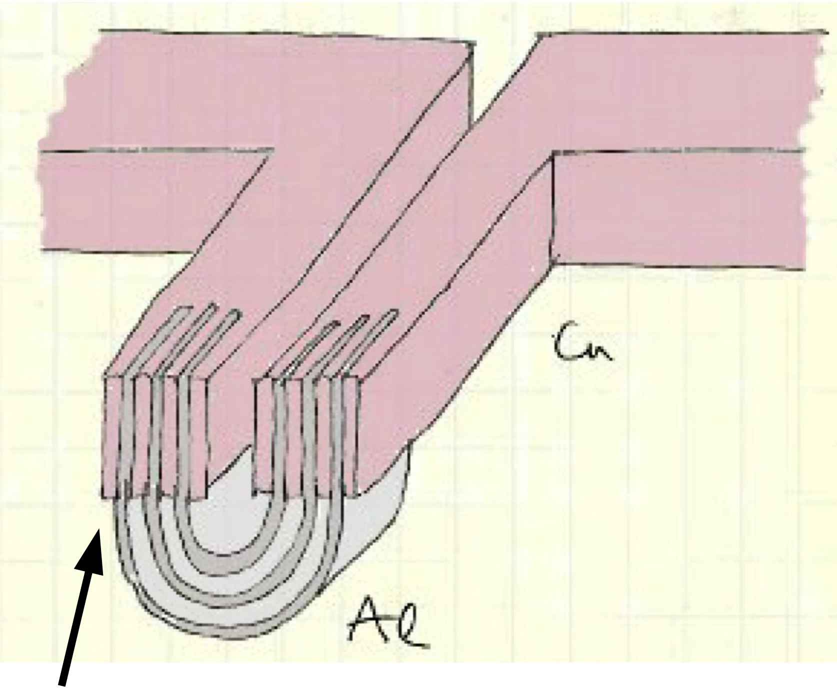
   Mechanical swiches may be used in pulse tube coolers

2) Precooling/isolation of single shot cooling stages
   - – demagnetization cooling (conventional or nuclear)

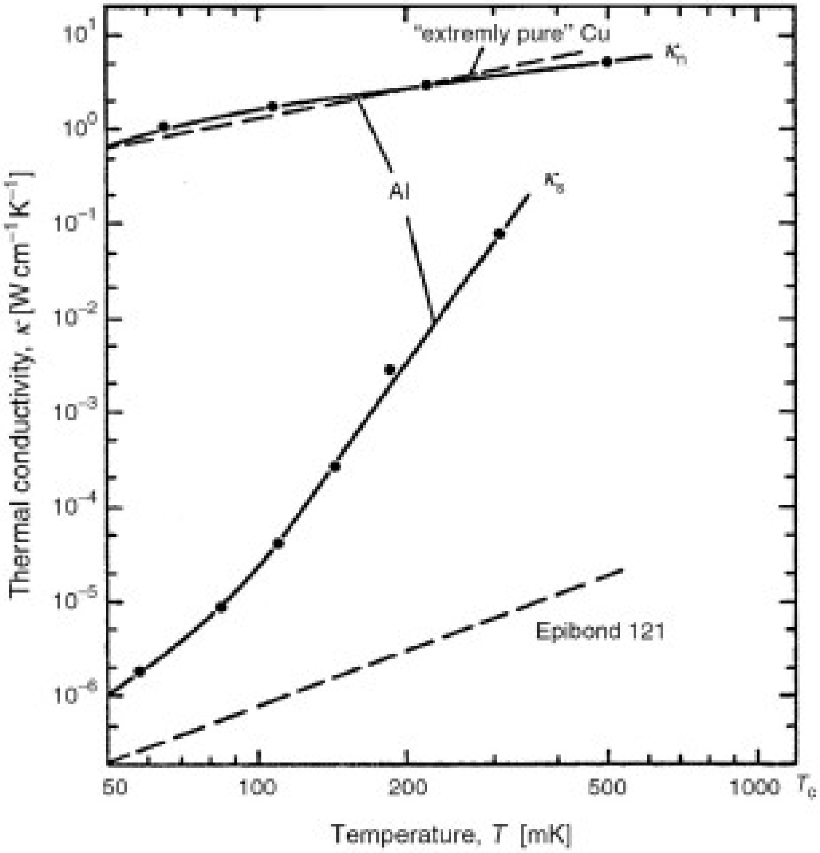
   Superconducting heat switch
   - – effective below $T_c/T \sim 10$ (usually < 0.1 K)
   - – switching ratio $> 10^6$ with a proper design

Superconducting heat switch

Switching element can be Al, In, Sn, Zn, Pb, …
($T_c \sim 1$ K, $B_c \sim 10 \ldots 100$ mT)



Diffusion welding



Kapitza resistance

Thermal coupling between liquid helium and solid substances

1) helium as the cooling agent at mK-temperatures
2) helium as the object of study down to ~ 0.1 mK
   - heat is carried across the boundary by phonons
   - phonons don't easily cross the interface (acoustic mismatch)
   - critical angle due to Snell's law: $\alpha_c = \arcsin v_h/v_s \sim 3^{\mathrm{o}}$
   - phonons within the critical cone: $f = \sin^2\alpha_c/2 \sim 10^{-3}$
   - power transmission is obtained as (acoustic impedance $z = \rho v$):

$$t = 4\, z_h z_s/(z_h+z_s)^2 \sim 4\, \rho_h v_h/\rho_s v_s \sim 2\cdot 10^{-3}$$

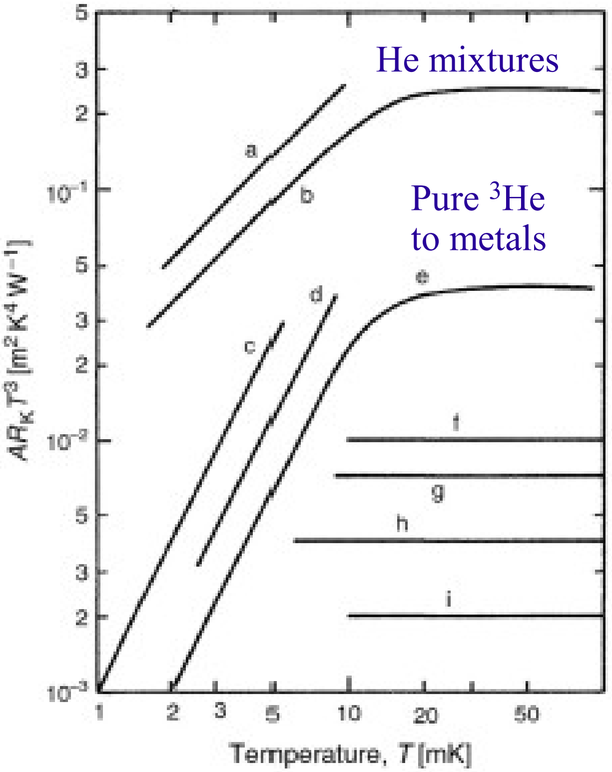
=> fraction of phonon energy getting through: $f\,t < 10^{-5}$

Kapitza resistance (cont.)

Debye theory gives the energy flux

$$\frac{\dot{Q}}{A} = \frac{\pi^2 k_B^4 T^4 \rho_h v_h}{30\hbar^3 \rho_s v_s^3}$$

Resulting in the Kapitza resistance (for $\Delta T << T$)



He mixtures

Pure ^{3}He to metals

^{3}He to some plastics

$$R_K = \frac{\Delta T}{\dot{Q}} = \frac{15\hbar^3 \rho_s v_s^3}{2\pi^2 k_B^4 T^3 A \rho_h v_h} \propto \frac{1}{AT^3}$$

Valid for $T \sim 10 \ldots 200$ mK

Kapitza resistance (cont.)

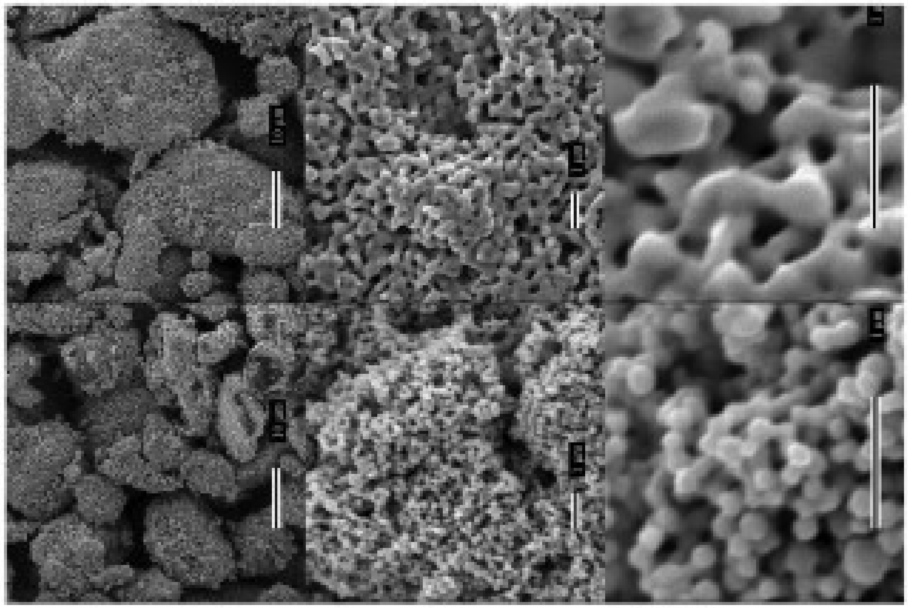
Use fine silver powder sintered on copper or silver surface to significantly increase the surface area (→ tens of m^2)

Particle size 40 … 1000 nm
Compressed by factor ~ 0.5
Heated gently (160 … 200 °C)
=> typically ~ 2 m^2/g

Not more than ~ 1 … 2 mm thick layer!

Bad area/vol; good conductivity

m2008.05.07
7g, D 25 mm x 3.3 mm, 500 kg
--- 200 C, 20 min --->
D 23.8 mm x 3.1 mm
0.49 m^2/g



Good area/vol; enough cond?

m2008.05.13
7g, D 25 mm x 3.3 mm, 500 kg
--- 155 C, 10 min --->
D 24.8 mm x 3.2 mm
2.06 m^2/g

Thermal expansion

Shrinkage is substantial (RT … 4K):
- dielectrics 1 … 2 %
- metals 0.2 … 0.4 %

Some special compounds have very small expansion/contraction
- Duran/Pyrex glass
- Invar steel

Most shrinkage occurs from RT to LN2 temperature

Inportant to stress test leak tight joints & electrical contacts by thermal cycling between RT & LN2 before actual use

Pay attention to proper order when making pressed joints or tubular connections from dissimilar materials

Steel – copper – brass from small shrinkage to large

“Black Stycast” epoxy (2850) has smallish expansion coefficient
=> good for casting on metals

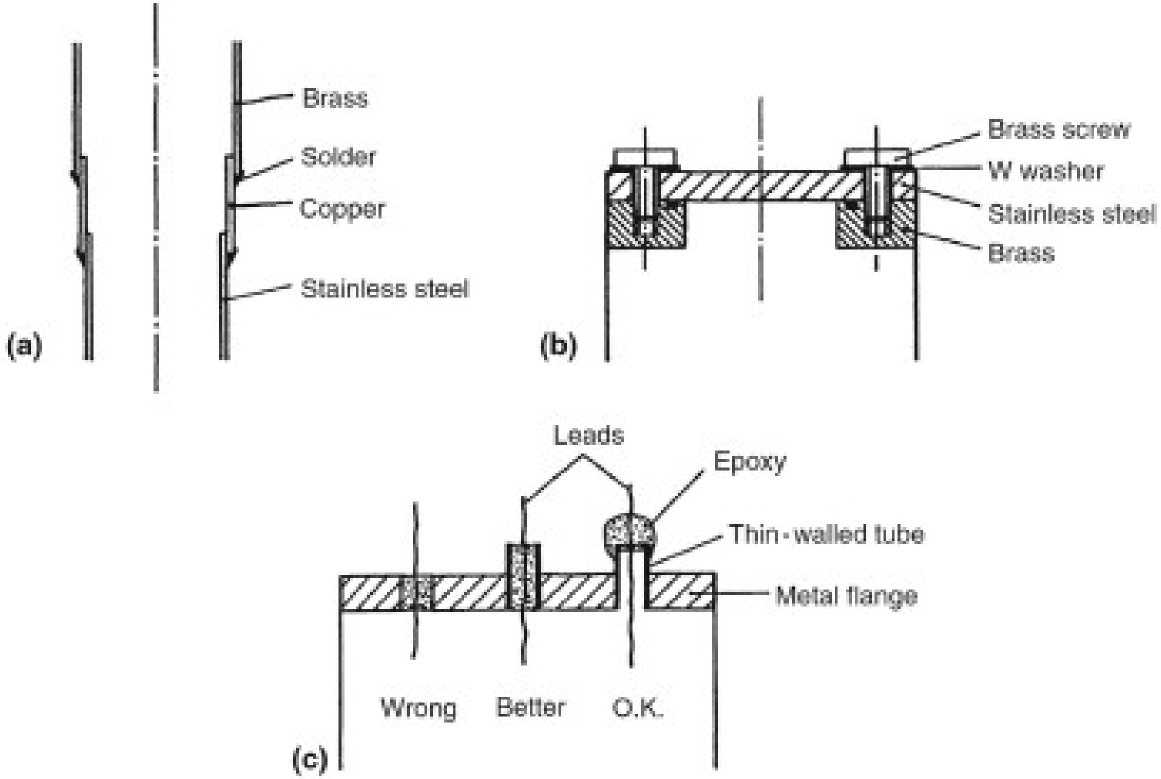


Fig. 3.18. When joining different materials in a cryogenic apparatus one has to take into account the difference in their thermal expansion coefficients. For example: (**a**) The tube with the largest expansion coefficient should be on the outside so that the solder joint is not pulled open during cooldown. (**b**) In an O-ring seal the screw should have a larger expansion coefficient than the flange. The seal will tighten even further during cooldown if a washer with a very small expansion coefficient is used. (**c**) In an epoxy feedthrough for leads the epoxy, with its large expansion coefficient, should contract on a thin-walled metal tube during cooldown rather than pull away. It helps if the tube walls are tapered to a sharp edge. An epoxy with filler should be used to lower its thermal expansion coefficient

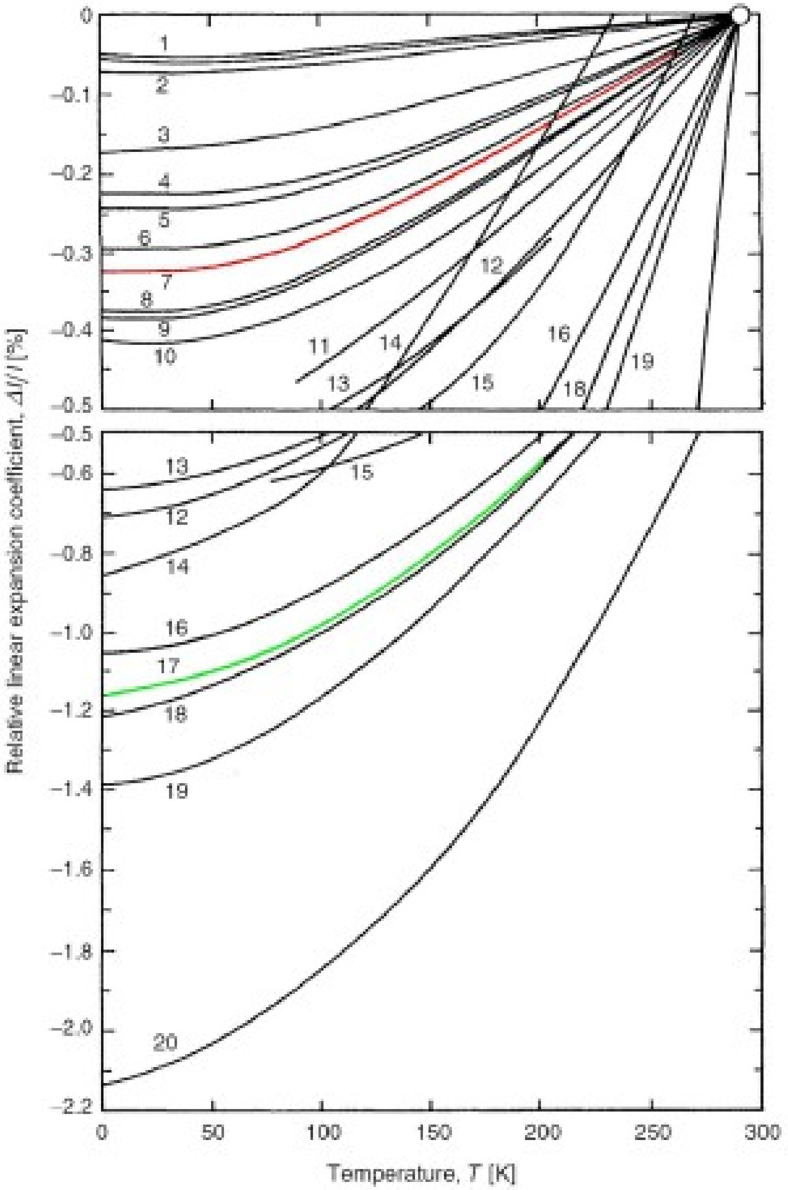


Fig. 3.17. Relative linear thermal expansion coefficient of (1) Invar (upper), Pyrex (lower), (2) W, (3) nonalloyed steel, (4) Ni, (5) $Cu_{0.7}Ni_{0.3}$, (6) stainless steel, (7) Cu, (8) German silver, (9) brass, (10) Al, (11) soft solder, (12) In, (13) Vespel SP22, (14) Hg, (15) ice, (16) Araldite, (17) Stycast 1266, (18) PMMA, (19) Nylon, (20) Teflon [3.76]. Some further data are: Pt similar to (3); Ag between (9) and (10); Stycast 2850 GT slightly larger than (10). The relative change of length between 300 and 4 K is $10^3 \Delta l/l = 12, 11.5, 4.4, 6.3$ and 5.7 for Polypropylene, Stycast 1266, Stycast 2850 GT as well as 2850 FT, Vespel SP-22 and solders, respectively [3.44, 3.55, 3.56, 3.76–3.82, 3.114]. Torlon behaves very similar to Stycast 2850FT [3.114]