Introduction to Superconductivity

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CONTENTS

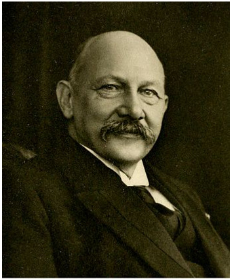
QUANTUM MECHANICS AT A HUMAN SCALE

Superconductivity was the first discovered macroscopic quantum phenomenon, where a sizable fraction of particles of a macroscopic object forms a coherent state, described by a quantum-mechanical wave function.

Applications:

- High magnetic fields
- Filtering of radio signals
- Ultra-sensitive magnetic/temperature and other sensors
- Metrology
- Ultra-low-noise amplifiers
- Quantum engineering
- Quantum information processing

DISCOVERY OF SUPERCONDUCTIVITY



Heike Kamerlingh Onnes

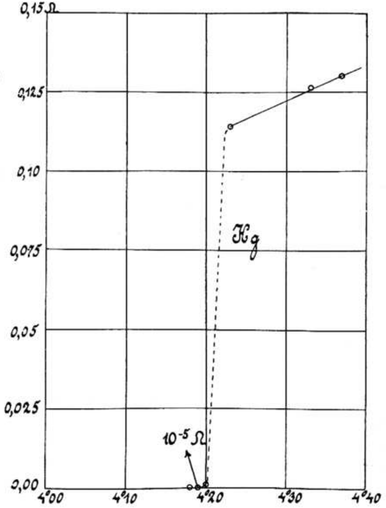
Liquefied helium in 1908.

Resistivity of metals when $T \to 0$?

Au, Pt: $\rho(T) = \text{const}$

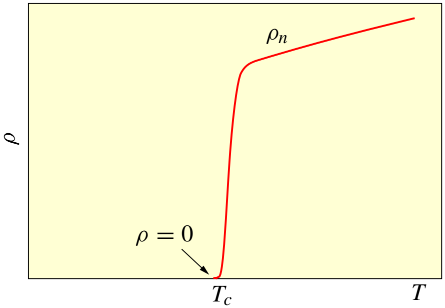
Mercury sample, 1911

Resistance, Ohm



Temperature, K

BASIC EXPERIMENTAL PROPERTIES: Zero resistivity



Below the transition temperature T_c, the resistivity drops to zero.

Latest measurements put bound $\rho < 10^{-23}\,\Omega\,\text{cm}$.

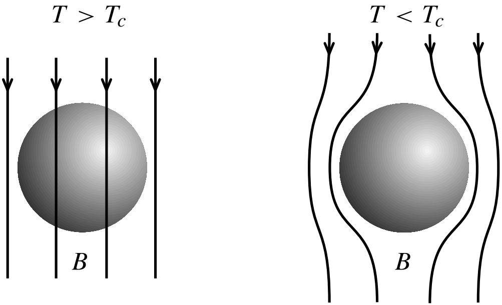
Really zero!

Dirty metals are good superconductors.

THE MEISSNER EFFECT

Walther Meissner

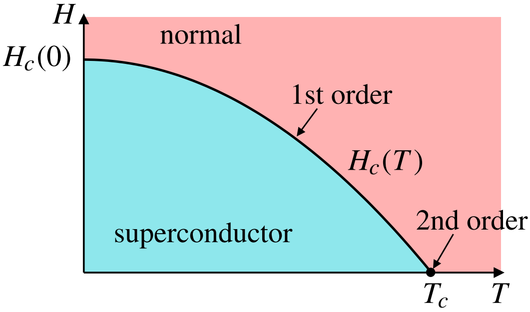
Robert Ochsenfeld



At $T < T_c$ the magnetic field is expelled from the superconductor even if the field was applied before reaching T_c.

PHASE DIAGRAM OF A SUPERCONDUCTOR

Superconductivity is destroyed by sufficiently strong electric currents or by the magnetic field above the critical field H_c.

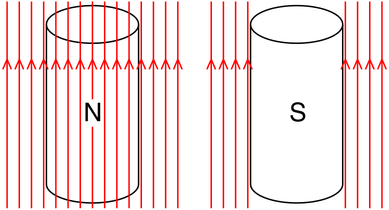


Empirically the dependence of H_c on temperature is described reasonably well by

$$H_c(T) = H_c(0)[1 - (T/T_c)^2]$$

THERMODYNAMICS OF THE SUPERCONDUCTING TRANSITION

In the external field $H = H_c$



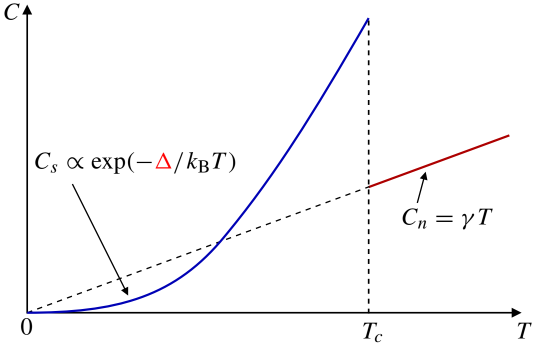
$$F_n(T) = F_s(T) + \frac{H_c^2(T)}{8\pi}$$

Entropy $S = -\partial F/\partial T$: $$S_n - S_s = -\frac{H_c}{4\pi}\frac{dH_c}{dT}.$$

Heat capacity $C = T\partial S/\partial T$: $$C_n - C_s = -\frac{T}{4\pi}\left[H_c\frac{d^2H_c}{dT^2} + \left(\frac{dH_c}{dT}\right)^2\right].$$

The heat capacity jump at T_c: $$C_s(T_c) - C_n(T_c) = \frac{T_c}{4\pi}\left(\frac{dH_c}{dT}\right)^2.$$

HEAT CAPACITY



Heat capacity jumps upward on transition to superconducting state. At $T \ll T_c$ the electronic heat capacity approaches zero exponentially $\Rightarrow$

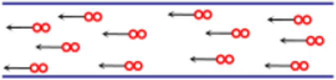
Energy gap Δ

.

PHASE COHERENCE

Electrons form pairs, which are bosons and condense to a single state, described by the wave function ψ with the macroscopically coherent phase χ.

$$\psi = |\psi| \exp(i\chi)$$



$$\Longrightarrow \mathbf{j}_s$$

Current

$$\mathbf{j}_s = \frac{e^*}{2m^*}\left[\psi^*\hat{\mathbf{p}}\psi + \psi\hat{\mathbf{p}}^\dagger\psi^*\right]$$

with

$$\hat{\mathbf{p}} = -i\hbar\nabla - (e^*/c)\mathbf{A}\,.$$

We get for $e^* = 2e$, $m^* = 2m$ and the density of pairs $|\psi|^2 = n_s/2$:

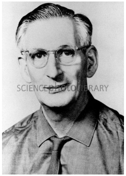
$$\mathbf{j}_s = -\frac{(e^*)^2}{m^*c}|\psi|^2\left(\mathbf{A} - \frac{\hbar c}{e^*}\nabla\chi\right) = -\frac{e^2 n_s}{mc}\left(\mathbf{A} - \frac{\hbar c}{2e}\nabla\chi\right)\,.$$

THE LONDON EQUATION

Assuming density of superconducting electrons n_s is constant:

$$\mathbf{j}_s = -\frac{e^2 n_s}{mc}\left(\mathbf{A} - \frac{\hbar c}{2e}\nabla\chi\right) \quad \Rightarrow \quad \text{curl}\,\mathbf{j}_s = -\frac{e^2 n_s}{mc}\,\text{curl}\,\mathbf{A} = -\frac{e^2 n_s}{mc}\,\mathbf{h}\,.$$

Fritz London



Heinz London

Using the Maxwell equation

$$\mathbf{j}_s = (c/4\pi)\,\text{curl}\,\mathbf{h}$$

we get the London equation

$$\mathbf{h} + \lambda_L^2\,\text{curl}\,\text{curl}\,\mathbf{h} = 0\,,$$

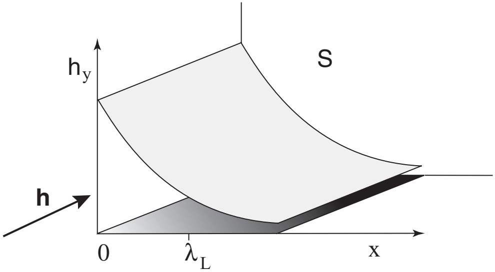
where the London penetration depth

$$\lambda_L = \left(\frac{mc^2}{4\pi n_s e^2}\right)^{1/2}\,.$$

MEISSNER EFFECT

$$\mathbf{h} + \lambda_L^2 \,\mathrm{curl}\,\mathrm{curl}\,\mathbf{h} = 0 \quad \Rightarrow \quad \mathbf{h} = \lambda_L^2 \nabla^2 \mathbf{h}$$

$$[\mathrm{curl}\,\mathrm{curl}\,\mathbf{h} = \nabla\, \cancel{\mathrm{div}\,\mathbf{h}} - \nabla^2 \mathbf{h}]$$



$$\frac{\partial^2 h_y}{\partial x^2} - \lambda_L^{-2} h_y = 0$$

$$h_y = h_y(0)\exp(-x/\lambda_L)$$

Magnetic field penetrates into a superconductor only over distances

$$\lambda_L = \left(\frac{mc^2}{4\pi n_s e^2}\right)^{1/2}$$

Typical metal with $m \sim m_e$ and $a_0 \sim 4\,\text{Å}$, $n_s \sim a_0^{-3}$ $\Rightarrow$ $\lambda_L \sim 30\,\mathrm{nm}$

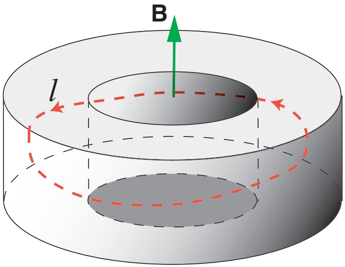
MAGNETIC FLUX QUANTIZATION

Integrating expression

$$\mathbf{j}_s = -\frac{e^2 n_s}{mc}\left(\mathbf{A} - \frac{\hbar c}{2e}\nabla\chi\right)$$

along a closed contour within a superconductor

$$-\frac{mc}{e^2}\oint n_s^{-1}\mathbf{j}_s\cdot d\mathbf{l} = \int_S \mathrm{curl}\,\mathbf{A}\cdot d\mathbf{S} - \frac{\hbar c}{2e}\Delta\chi = \Phi - \frac{\hbar c}{2e}2\pi n$$



Φ is the magnetic flux through the contour.

$$\Phi' = \Phi + \frac{4\pi}{c}\oint \lambda_L^2\,\mathbf{j}_s\cdot d\mathbf{l} = \Phi_0 n$$

Quantum of magnetic flux

$$\Phi_0 = \frac{\pi\hbar c}{|e|} \approx 2.07\times 10^{-7}\ \mathrm{Oe}\cdot\mathrm{cm}^2\,.$$

In SI units, $\Phi_0 = \pi\hbar/|e| = 2.07\times 10^{-15}\mathrm{T}\cdot\mathrm{m}^2$.

THE ENERGY GAP AND COHERENCE LENGTH

Energy scale: Energy gap Δ_0

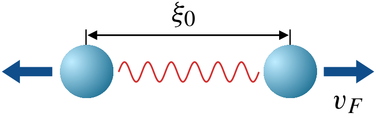
Binding energy of Cooper pair $2\Delta_0$, $\Delta_0 \sim k_{\mathrm{B}} T_c$.

Energy gap (from the heat capacity) $\Delta(T \to 0) = \Delta_0$.

Uncertainty principle: interaction time in the pair $\tau_p \gtrsim \hbar/\Delta_0$.

Length scale: Coherence length ξ_0:

$$\xi_0 \sim \tau_p v_F \sim \frac{\hbar v_F}{\Delta_0}$$



COHERENCE LENGTH AND THE CRITICAL FIELD

Maximum phase gradient $(\nabla\chi)_{\max} \sim 1/\xi$ sets the maximum supercurrent

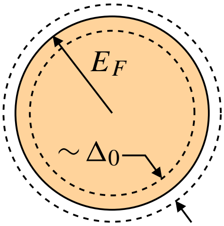
$$\mathbf{j}_s = -\frac{e^2 n_s}{mc}\left(\mathbf{A} - \frac{\hbar c}{2e}\nabla\chi\right) \quad \Rightarrow \quad (j_s)_{\max} \sim \frac{\hbar n_s e}{m}(\nabla\chi)_{\max}$$

Maximum supercurrent is reached when screening the critical field H_c

$$(j_s)_{\max} \sim (c/4\pi) H_c/\lambda_L$$

Critical field

$$H_c \sim \frac{4\pi\lambda_L}{c}\frac{\hbar n_s e}{m\xi} = \frac{\pi\hbar c}{e}\frac{\lambda_L}{\pi\xi}\frac{4\pi n_s e^2}{mc^2} = \frac{\Phi_0}{\pi\xi\lambda_L}$$



Condensation energy [remember $\xi_0 \sim \hbar v_F/\Delta_0$]

$$F_n(0) - F_s(0) = \frac{H_c^2(0)}{8\pi} \sim \frac{n}{2mv_F^2}\Delta_0^2 \sim [N(0)\Delta_0]\Delta_0$$

$N(0) \sim n/E_F$ — density of states, $E_F = mv_F^2/2$ — Fermi energy

LONDON AND PIPPARD REGIMES

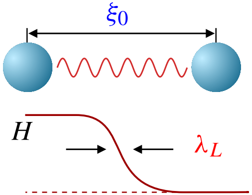
Two length scales: London penetration depth and coherence length.

How do their values compare?

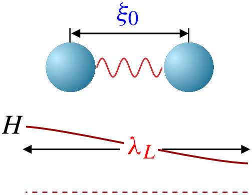
Typical metallic superconductor [like Al] with $T_c = 1\,\mathrm{K}$, the electron density n one per ion, lattice constant $a_0 \sim 4\,\text{Å}$:

$$n_s \approx n = a_0^{-3} \approx 4 \cdot 10^{22}\,\mathrm{cm}^{-3}, \;\; v_F = p_F/m = (\hbar/m)(3\pi^2 n)^{1/3} \approx 10^8\,\mathrm{cm/s}$$

$$\xi_0 = \frac{\hbar v_F}{2\pi k_B T_c} \approx 1400\,\mathrm{nm} \;\gg\; \lambda_L = \left(\frac{mc^2}{4\pi n_s e^2}\right)^{1/2} \approx 30\,\mathrm{nm}$$



Pippard regime



London regime

$T \to T_c$

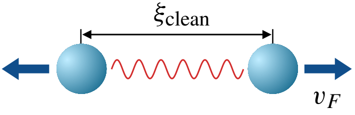
$n_s \to 0$

$\lambda_L \to \infty$

DIRTY AND CLEAN LIMITS

Non-magnetic impurities (scattering centers) do not affect static properties of a superconductor (like T_c) [Anderson theorem].

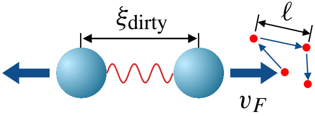
But properties connected to the spatial variation of the superconducting state (in particular, supercurrents and coherence length) are strongly affected.



Interaction time $\tau_p \sim \hbar/\Delta_0$.

Clean limit – ballistic motion

$$\xi_{\text{clean}} = \xi_0 \sim \tau_p v_F \sim \frac{\hbar v_F}{\Delta_0}$$



Dirty limit – diffusive motion

mean free path ℓ

scattering time $\tau_s = \ell/v_F < \tau_p$

diffusion coefficient $D \sim v_F^2 \tau_s = v_F \ell$

$$\xi_{\text{dirty}} = \sqrt{D\tau_p} = \sqrt{v_F \ell \hbar/\Delta_0} = \sqrt{\xi_{\text{clean}} \ell}$$

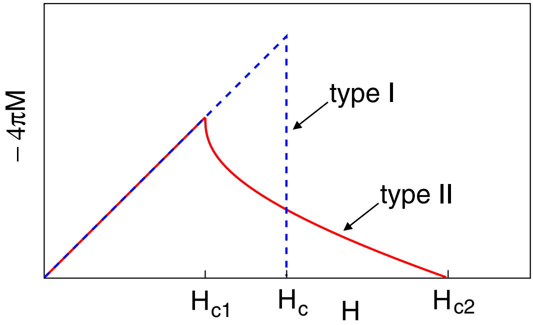
In dirty materials $\ell \sim a_0 \ll \lambda_L$ and $\xi_{\text{dirty}} < \lambda_L$

TYPE I AND TYPE II SUPERCONDUCTORS

In bulk superconductor magnetic induction $\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M} = 0.$

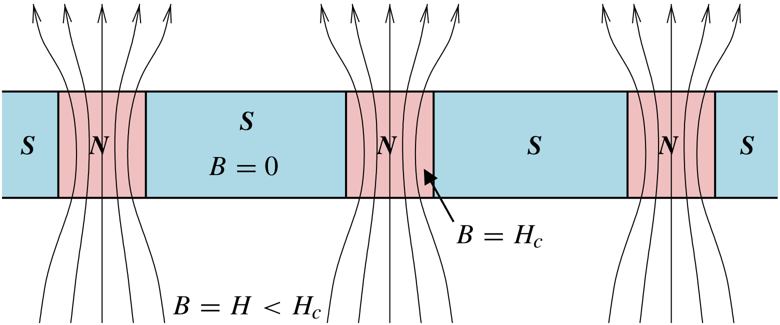
The magnetization and susceptibility are ideal diamagnetic

$$\mathbf{M} = -\frac{\mathbf{H}}{4\pi}\,;\;\; \chi = \frac{\partial M}{\partial H} = -\frac{1}{4\pi}$$

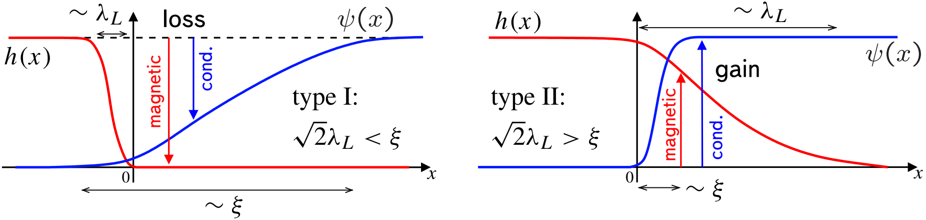


INTERMEDIATE STATE OF TYPE I SUPERCONDUCTORS

In the external field $H < H_c$ the sample is divided to normal and superconducting domains so that in the normal phase magnetic induction $B = H_c$ while in the superconducting domains magnetic field is absent.



ENERGY OF THE NS BOUNDARY



Energy change compared to the uniform state at $H = H_c$

$$\Delta F = \frac{H_c^2}{8\pi}\,\delta : \qquad \begin{array}{lll} \xi \gg \lambda_L & \delta \sim \xi - \lambda_L > 0 & \delta \approx 1.89\xi \\ \xi \ll \lambda_L & \delta \sim -(\lambda_L - \xi) < 0 & \delta \approx -1.104\lambda_L \end{array}$$

Transition from the positive to the negative energy of the NS boundary is controlled by the Ginzburg-Landau parameter κ:

$$\kappa = \frac{\lambda_L}{\xi}, \qquad \text{transition at } \kappa = \frac{1}{\sqrt{2}}\,.$$

THE GL PARAMETER FROM MATERIAL PARAMETERS

Order-parameter variation scale – coherence (healing) length $\xi(T)$ is *not* the Cooper-pair size ξ_0 (or $\xi_{\text{dirty}} = \sqrt{\xi_0 \ell}$):

$\xi(T)$: gradient energy $[\xi_0] \leftrightarrow$ condensation energy [T-dependent]

In the clean limit we have

$$\xi(T) = \xi_0 \sqrt{\frac{7\zeta(3)}{12}} \left[1 - \frac{T}{T_c}\right]^{-1/2}, \quad \lambda_L = \frac{c}{4|e|v_F} \sqrt{\frac{3}{\pi N(0)}} \left[1 - \frac{T}{T_c}\right]^{-1/2}$$

$$\kappa = \frac{3c}{|e|\hbar} \sqrt{\frac{\pi}{7\zeta(3) N(0)}} \frac{k_B T_c}{v_F^2} = \frac{3\pi^2}{\sqrt{14\zeta(3)}} \frac{\hbar c}{e^2} \frac{k_B T_c}{E_F} \sqrt{\frac{e^2/a_0}{E_F}} \sim 10^3 \frac{k_B T_c}{E_F}.$$

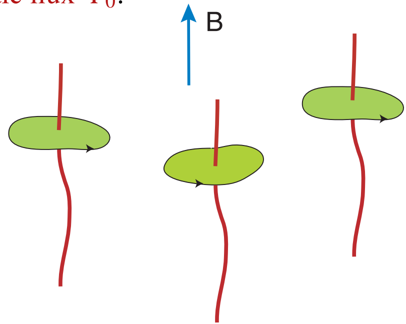
For usual superconductors	$k_B T_c / E_F < 10^{-3}$	$\Rightarrow$	$\kappa \lesssim 1$
For HTSC	$k_B T_c / E_F \sim 10^{-1} - 10^{-2}$	$\Rightarrow$	$\kappa \gg 1$

For dirty superconductors with $\tau_s/\tau_p \ll 1$: $\kappa_{\text{dirty}} \sim \kappa_{\text{clean}}(\tau_p/\tau_s) \gg 1$

ABRIKOSOV VORTICES

Magnetic field penetrates into type II superconductor in the form of Abrikosov vortices, which are topologically-protected linear defects of the order parameter: Order parameter is zero at the vortex axis and the phase of the order parameter winds by 2π on a loop around the vortex. Each vortex carries a single quantum of magnetic flux Φ_0.

Alexei Abrikosov



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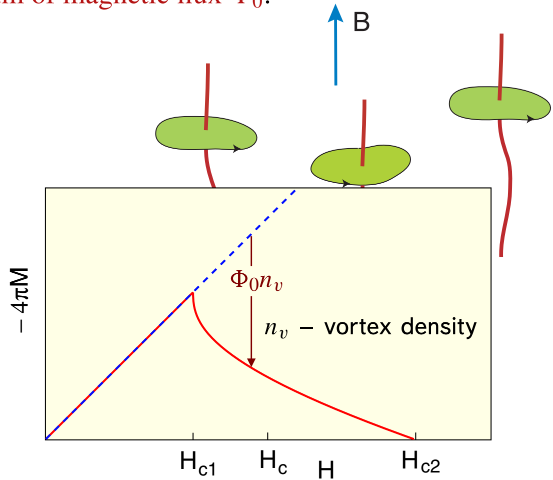
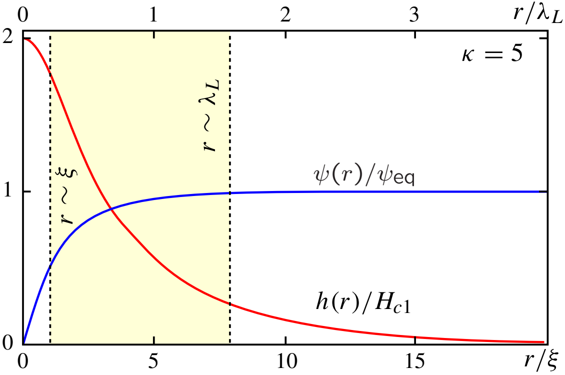


ABRIKOSOV VORTEX STRUCTURE



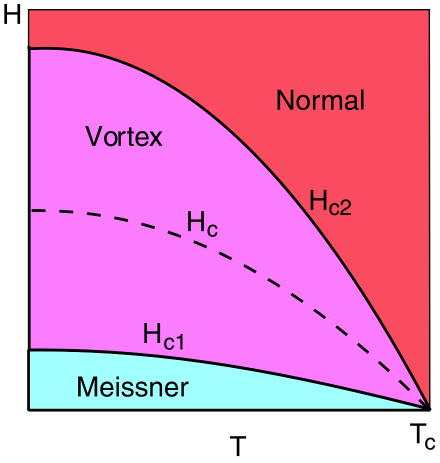
$r > \lambda_L$: $\quad f = \psi(r)/\psi_{\rm eq} = 1, \quad h \propto \exp(-r/\lambda_L)$

$\xi < r < \lambda_L$: $\quad 1 - f \propto r^{-2}, \quad h \propto \ln(\lambda_L/r)$

Main vortex energy here: $\mathcal{F}_v \approx \dfrac{\Phi_0^2}{16\pi^2\lambda_L^2} \ln\kappa$

$r < \xi$ (core): $\quad f \propto r, \quad h \approx \text{const}$

PHASE DIAGRAM OF TYPE-II SUPERCONDUCTORS



H_{c1}: vortex energetically favorable

$$\mathcal{F}_v = (1/4\pi)\Phi_0 H_{c1}$$

$$H_{c1} = \frac{\Phi_0}{4\pi\lambda_L^2}\ln\kappa = H_c\frac{\ln\kappa}{\sqrt{2}\kappa}$$

H_c: thermodynamic

$$H_c = \frac{\Phi_0}{2\sqrt{2}\pi\lambda_L\xi}$$

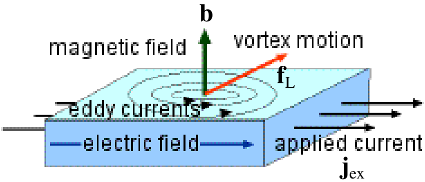
H_{c2}: no stable SC regions

$$H_{c2} = \frac{\Phi_0}{2\pi\xi^2} = H_c\sqrt{2}\kappa$$

For $\xi \sim 2\,\mathrm{nm}$ one gets $H_{c2} \sim 80\,\mathrm{T}$.

RESISTIVITY FROM VORTEX MOTION

Lorentz force acting on vortex from electric current



$$\mathbf{f}_L = \frac{\Phi_0}{c}[\mathbf{j}^{\mathrm{ex}} \times \mathbf{b}]\,.$$

Per unit volume we have

$$\mathbf{F}_L = n_v \mathbf{f}_L = c^{-1}[\mathbf{j}^{\mathrm{ex}} \times (n_v \Phi_0 \mathbf{b})] = c^{-1}[\mathbf{j}^{\mathrm{ex}} \times \mathbf{B}]\,.$$

If vortex moves with friction η: $\mathbf{v}_L = \mathbf{f}_L/\eta$

$$\mathbf{E} = c^{-1}[\mathbf{B} \times \mathbf{v}_L] = (\Phi_0 B/\eta c^2)\mathbf{j}^{\mathrm{ex}}$$

and resistivity

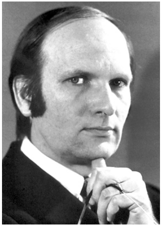
$$\rho = E/j^{\mathrm{ex}} = \Phi_0 B/\eta c^2\,.$$

Magnetic field applications require pinning!

The BCS Theory

John Bardeen

Leon Cooper



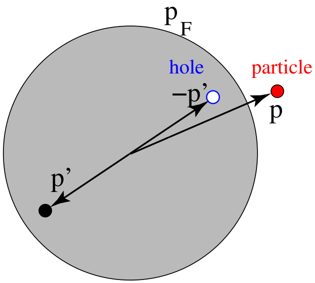
John Robert Schrieffer

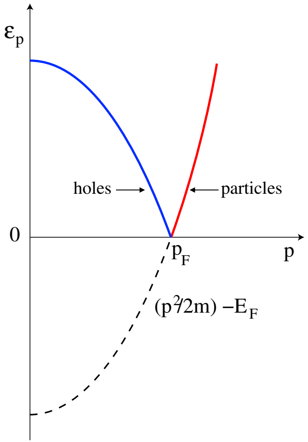
Nikolai Bogolubov

EXCITATIONS IN LANDAU FERMI LIQUID

At $T = 0$ states with $p < p_F$, $E < E_F$ are filled, others are empty.

$$n = \frac{p_F^3}{3\pi^2\hbar^3}, \qquad p_F \sim \hbar n^{1/3} = \hbar/a_0$$





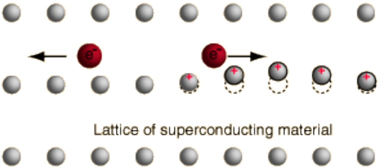
$$\epsilon_{\mathbf{p}} = \begin{cases} \frac{p^2}{2m} - E_F, & p > p_F \\ E_F - \frac{p^2}{2m}, & p < p_F \end{cases}$$

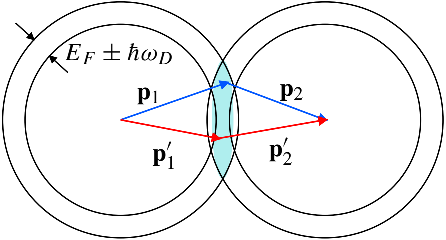
$$\epsilon_{\mathbf{p}} \approx v_F |p - p_F|$$

$$f(\epsilon_{\mathbf{p}}) = \frac{1}{e^{\epsilon_{\mathbf{p}}/k_B T} + 1}$$

Constant density of states $N(\epsilon_{\mathbf{p}} = 0) = N(p = p_F) = N(0) = \dfrac{m^* p_F}{2\pi^2\hbar^3}$.

ELECTRON ATTRACTION: Polarization of the lattice



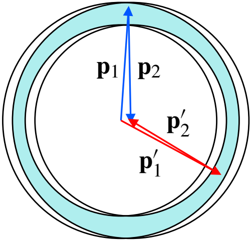


Atomic energy scale

$$M\omega_D^2 a_0^2 \sim \frac{e^2}{a_0} \sim \frac{(\hbar/a_0)^2}{m}$$

Debye frequency

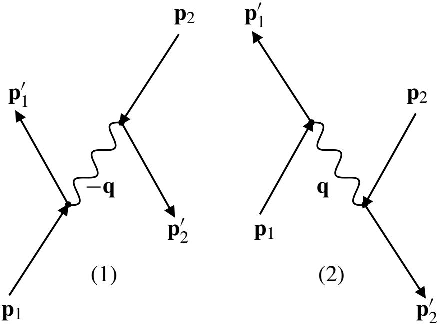
$$\omega_D \sim \frac{E_F}{\hbar}\sqrt{\frac{m}{M}}$$



Length of electron tail

$$\sim v_F \omega_D^{-1} \sim \sqrt{M/m}\, a_0 \sim 300 a_0 \quad \Rightarrow \quad \mathbf{p}_1 \approx \pm \mathbf{p}_2 \quad \Rightarrow \mathbf{p}_1 \approx -\mathbf{p}_2$$

ELECTRON ATTRACTION: Model



$$\langle \mathrm{II}|H_{\mathrm{e-ph-e}}|\mathrm{I}\rangle = \frac{2|W_{\mathbf{q}}|^2}{\hbar}\frac{\omega_{\mathbf{q}}}{\omega^2-\omega_{\mathbf{q}}^2} < 0$$

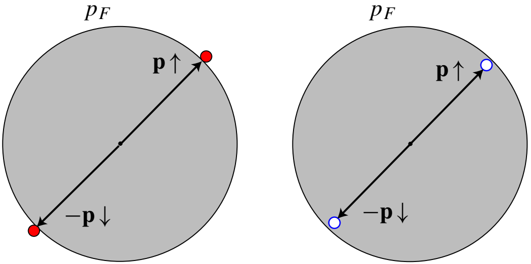
attraction if $\omega < \omega_{\mathbf{q}} \sim \omega_D$

Does not depend on the directions of $\mathbf{p}_1$, $\mathbf{p}_2 \Rightarrow$ orbital momentum $L = 0$

$\Rightarrow$ opposite spins

Model: Two electrons with opposite momenta and spins attract each other with the constant amplitude $-W$ if $\epsilon_{\mathbf{p}_1} < E_c$ and $\epsilon_{\mathbf{p}_2} < E_c$ (where $E_c \sim \hbar\omega_D$) and do not interact otherwise.

THE COOPER PROBLEM



Schrödinger equation for the pair wave function $\Psi(\mathbf{r}_1, \mathbf{r}_2)$

$$\left[\hat{H}_e(\mathbf{r}_1) + \hat{H}_e(\mathbf{r}_2) + W(\mathbf{r}_1 - \mathbf{r}_2)\right]\Psi(\mathbf{r}_1, \mathbf{r}_2) = E\Psi(\mathbf{r}_1, \mathbf{r}_2)\,.$$

Expansion in single-particle wave functions

$$\Psi(\mathbf{r}_1, \mathbf{r}_2) = \sum_{\mathbf{p}} c_{\mathbf{p}}\psi_{\mathbf{p}\uparrow}(\mathbf{r}_1)\psi_{-\mathbf{p}\downarrow}(\mathbf{r}_2) = \sum_{\mathbf{p}} a_{\mathbf{p}} e^{i\mathbf{pr}/\hbar} = \Psi(\mathbf{r})\,, \quad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2\,.$$

$$\psi_{\mathbf{p}\uparrow}(\mathbf{r}_1) \propto e^{i\mathbf{pr}_1/\hbar}, \quad \psi_{-\mathbf{p}\downarrow}(\mathbf{r}_2) \propto e^{-i\mathbf{pr}_2/\hbar}, \quad a_{\mathbf{p}} = \int \Psi(\mathbf{r})e^{-i\mathbf{pr}/\hbar}\, d^3\mathbf{r}$$

THE COOPER PROBLEM: Fourier transformed equation

Schrödinger equation becomes

$$2\epsilon_{\mathbf{p}} a_{\mathbf{p}} + \sum_{\mathbf{p}'} W_{\mathbf{p},\mathbf{p}'} a_{\mathbf{p}'} = E a_{\mathbf{p}} \, .$$

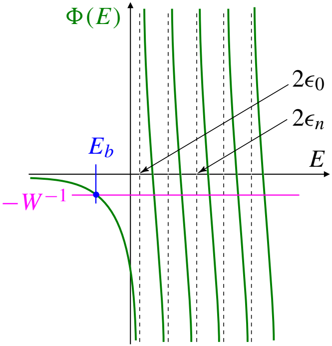
Interaction model:

$$W_{\mathbf{p},\mathbf{p}'} = \begin{cases} -W, & \epsilon_{\mathbf{p}} \text{ and } \epsilon_{\mathbf{p}'} < E_c, \\ & \text{i.e. } p_F - E_c/v_F < p (\text{and } p') < p_F + E_c/v_F \\ 0, & \text{otherwise} \end{cases}$$

We thus have

$$a_{\mathbf{p}} = -\frac{W}{E - 2\epsilon_{\mathbf{p}}} \sum_{\mathbf{p}', \epsilon_{\mathbf{p}'} < E_c} a_{\mathbf{p}'} = -\frac{W C}{E - 2\epsilon_{\mathbf{p}}}$$

$$-\frac{1}{W} = \sum_{\mathbf{p}, \epsilon_{\mathbf{p}} < E_c} \frac{1}{E - 2\epsilon_{\mathbf{p}}} \equiv \Phi(E)$$



THE COOPER PROBLEM: Bound state

Bound state with $E = E_b < 0$

$$\frac{1}{W} = \sum_{\mathbf{p}, \epsilon_{\mathbf{p}} < E_c} \frac{1}{2\epsilon_{\mathbf{p}} - E_b} \, .$$

Sum over momentum → the integral over energy:

$$\frac{1}{W} = 2N(0) \int_0^{E_c} \frac{d\epsilon_{\mathbf{p}}}{2\epsilon_{\mathbf{p}} + |E_b|} = N(0) \ln \left(\frac{|E_b| + 2E_c}{|E_b|} \right) \, .$$

From this equation we obtain

$$|E_b| = \frac{2E_c}{e^{1/N(0)W} - 1}$$

For weak coupling, $N(0)W \ll 1$, we find

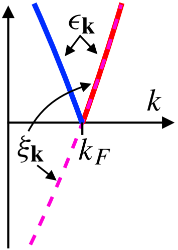
$$|E_b| = 2E_c e^{-1/N(0)W}$$

For strong coupling, $N(0)W \gg 1$,

$$|E_b| = 2N(0)WE_c$$

THE BCS MODEL: Non-interacting Hamiltonian

Non-interacting particles at $k > k_F$ and holes at $k < k_F$



$$H_0 = \sum_{\mathbf{k}\sigma, k>k_F} \epsilon_{\mathbf{k}} c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \sum_{\mathbf{k}\sigma, k<k_F} \epsilon_{\mathbf{k}} h^\dagger_{\mathbf{k}\sigma} h_{\mathbf{k}\sigma} \,.$$

But $h^\dagger_{\mathbf{k}\sigma} = c_{-\mathbf{k},-\sigma}$ and $h_{\mathbf{k}\sigma} = c^\dagger_{-\mathbf{k},-\sigma}$ and we have

$$\begin{aligned} H_0 &= \sum_{\mathbf{k}\sigma, k>k_F} \epsilon_{\mathbf{k}} c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \sum_{\mathbf{k}\sigma, k<k_F} \epsilon_{-\mathbf{k}} c_{\mathbf{k}\sigma} c^\dagger_{\mathbf{k}\sigma} \\ &= \sum_{\mathbf{k}\sigma, k>k_F} \epsilon_{\mathbf{k}} c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} - \sum_{\mathbf{k}\sigma, k<k_F} \epsilon_{\mathbf{k}} c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \sum_{\mathbf{k}\sigma, k<k_F} \epsilon_{\mathbf{k}} = \sum_{\mathbf{k}\sigma} \xi_{\mathbf{k}} c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + E_0 \,. \end{aligned}$$

$$c_{\mathbf{k}\sigma} c^\dagger_{\mathbf{k}\sigma} = 1 - c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma}$$

Here we defined

$$\xi_{\mathbf{k}} = \mathrm{sign}(k - k_F)\epsilon_{\mathbf{k}} = \frac{\hbar^2 k^2}{2m} - E_F \approx \hbar v_F (k - k_F) \,.$$

THE BCS MODEL: Pairing interaction

Interaction $(\mathbf{k}', -\mathbf{k}') \to (\mathbf{k}, -\mathbf{k})$ without affecting the quasiparticle spin.

$$H = \sum_{\mathbf{k}\sigma} \xi_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \sum_{\mathbf{k}\mathbf{k}'} W_{\mathbf{k}\mathbf{k}'} \underbrace{c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow}}_{A} \underbrace{c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow}}_{B}$$

Mean-field approximation: For two operators A and B

$$AB = \langle A \rangle B + A \langle B \rangle - \langle A \rangle \langle B \rangle + \cancel{(A - \langle A \rangle)(B - \langle B \rangle)}$$

The error is quadratic in fluctuations, which are relatively small in a macroscopic system.

THE BCS MODEL: Hamiltonian

$$H_{\rm BCS} = \sum_{\mathbf{k}\sigma} \xi_{\mathbf{k}} c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \sum_{\mathbf{k}} \left(\Delta^*_{\mathbf{k}} c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} + \Delta_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} - \Delta_{\mathbf{k}} \left\langle c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} \right\rangle \right)$$

We defined $\Delta_{\mathbf{k}} = \sum_{\mathbf{k}'} W_{\mathbf{k}\mathbf{k}'} \langle c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow} \rangle$, thus $\Delta^*_{\mathbf{k}} = \sum_{\mathbf{k}'} W_{\mathbf{k}\mathbf{k}'} \left\langle c^\dagger_{\mathbf{k}'\uparrow} c^\dagger_{-\mathbf{k}'\downarrow} \right\rangle$

We want to diagonalize it with **Bogolubov transformation**

$$H_{\rm BCS} = \sum_{\mathbf{k}\sigma} E_{\mathbf{k}} \gamma^\dagger_{\mathbf{k}\sigma} \gamma_{\mathbf{k}\sigma} + E_{\rm cond}$$

Here $\gamma^\dagger_{\mathbf{k}\sigma}$ and $\gamma_{\mathbf{k}\sigma}$ are new Bogolubov quasiparticles with spectrum $E_{\mathbf{k}}$.

New operators mix particles and holes:

$$\gamma^\dagger_{\mathbf{k}\uparrow} = u_{\mathbf{k}}\, c^\dagger_{\mathbf{k}\uparrow} + v_{\mathbf{k}}\, h^\dagger_{\mathbf{k}\uparrow} \qquad\qquad h^\dagger_{\mathbf{k}\uparrow} = c_{-\mathbf{k}\downarrow}$$

$$\gamma_{-\mathbf{k}\downarrow} = u^*_{\mathbf{k}}\, c_{-\mathbf{k}\downarrow} - v^*_{\mathbf{k}}\, h_{-\mathbf{k}\downarrow} \qquad\qquad h_{-\mathbf{k}\downarrow} = c^\dagger_{\mathbf{k}\uparrow}$$

$$\left\{ \gamma_{\mathbf{k}\sigma}, \gamma^\dagger_{\mathbf{k}'\sigma'} \right\} = \delta_{\mathbf{k}\mathbf{k}'}\delta_{\sigma\sigma'}, \ \ \{\gamma_{\mathbf{k}\sigma}, \gamma_{\mathbf{k}'\sigma'}\} = \left\{ \gamma^\dagger_{\mathbf{k}\sigma}, \gamma^\dagger_{\mathbf{k}'\sigma'} \right\} = 0 \quad \Rightarrow \quad |u_{\mathbf{k}}|^2 + |v_{\mathbf{k}}|^2 = 1$$

THE BCS MODEL: Diagonal form

The desired form is

$$H_{\mathrm{BCS}} = \sum_{\mathbf{k}\sigma} E_{\mathbf{k}} \gamma^{\dagger}_{\mathbf{k}\sigma} \gamma_{\mathbf{k}\sigma} + E_{\mathrm{cond}}$$

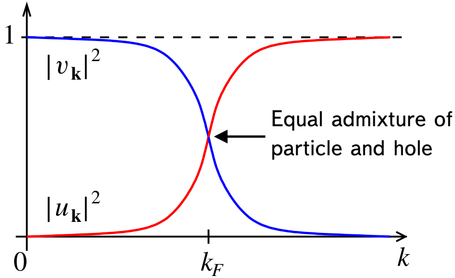
with

$$E_{\mathbf{k}} = \sqrt{\xi_{\mathbf{k}}^2 + |\Delta_{\mathbf{k}}|^2}$$

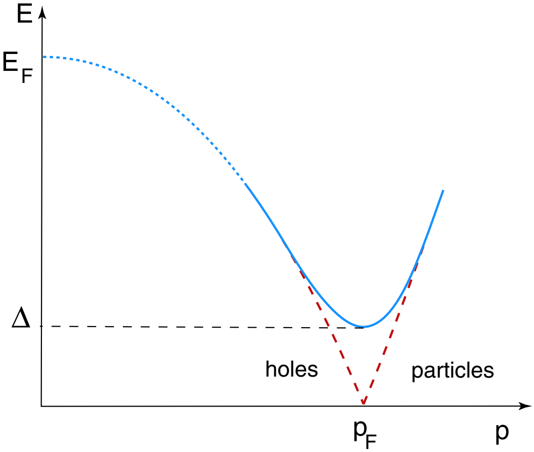
and

$$|u_{\mathbf{k}}|^2 = \frac{1}{2}\left(1 + \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}\right)$$

$$|v_{\mathbf{k}}|^2 = \frac{1}{2}\left(1 - \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}\right)$$



BOGOLUBOV QUASIPARTICLES: Energy spectrum



$$E_{\mathbf{p}} = \sqrt{\epsilon_{\mathbf{p}}^2 + |\Delta|^2}$$

$$\approx \sqrt{v_F^2 (p - p_F)^2 + |\Delta|^2}$$

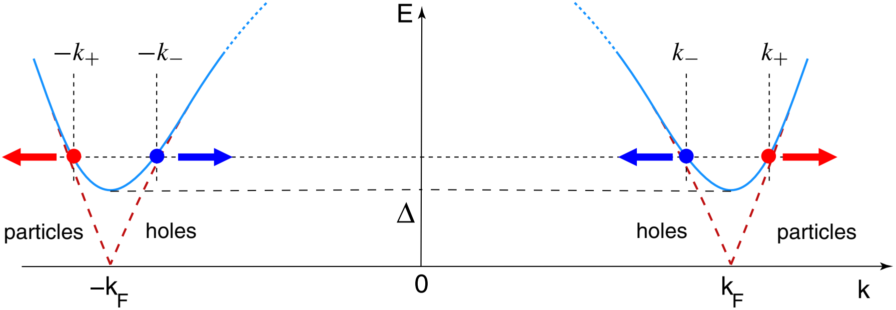
Landau criterion $v_c = \min(E_{\mathbf{p}}/p) \approx |\Delta|/p_F$

BOGOLUBOV QUASIPARTICLES: Group velocity

For a given energy $E > |\Delta|$ there are two possible values of $\xi_{\mathbf{k}}$:

$$\xi_{\mathbf{k}}^{\pm} = \pm\sqrt{E^2 - |\Delta|^2} = \hbar v_F (k_\pm - k_F)\,, \quad k_\pm = k_F \pm \frac{1}{\hbar v_F}\sqrt{E^2 - |\Delta|^2}\,.$$

$\xi_{\mathbf{k}}^{+}$, k_+ — particles, $\xi_{\mathbf{k}}^{-}$, k_- — holes



$$\mathbf{v}_g = \frac{dE}{d\mathbf{p}} = \frac{\hat{\mathbf{k}}}{\hbar}\frac{dE}{dk}\,, \quad \mathbf{v}_g^{\text{particles}} = v_F \frac{\sqrt{E^2 - |\Delta|^2}}{E}\hat{\mathbf{k}}, \quad \mathbf{v}_g^{\text{holes}} = -v_F \frac{\sqrt{E^2 - |\Delta|^2}}{E}\hat{\mathbf{k}}.$$

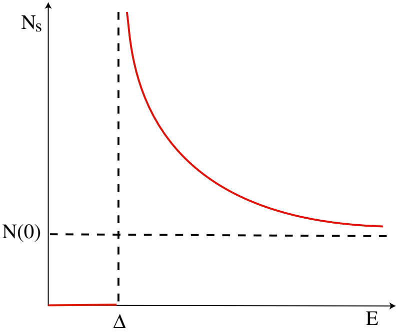
BOGOLUBOV QUASIPARTICLES: Density of states

Since $E_{\mathbf{k}} = \sqrt{\epsilon_{\mathbf{k}}^2 + |\Delta|^2}$ we have

$$\sum_{\mathbf{k}} \rightarrow N(0) \int d\epsilon_{\mathbf{k}} = N(0) \int d\left(\sqrt{E_{\mathbf{k}}^2 - |\Delta|^2}\right) = N(0) \int \frac{E_{\mathbf{k}} dE_{\mathbf{k}}}{\sqrt{E_{\mathbf{k}}^2 - |\Delta|^2}}.$$

The density of states in the superconductor is

$$N_s(E) = \begin{cases} 0, & E \leqslant |\Delta|, \\ N(0)\dfrac{E}{\sqrt{E^2 - |\Delta|^2}}, & E > |\Delta|. \end{cases}$$



SELF-CONSISTENCY EQUATION

We insert Bogolubov transformation $[c_{\mathbf{k}\sigma} \to \gamma_{\mathbf{k}\sigma} \to (u_{\mathbf{k}}, v_{\mathbf{k}}) \to (\Delta_{\mathbf{k}}, E_{\mathbf{k}})]$ into the gap definition and obtain equation for $\Delta_{\mathbf{k}}$ [remember also $E_{\mathbf{k}}(\Delta_{\mathbf{k}})$]

$$\Delta_{\mathbf{k}} = \sum_{\mathbf{k}'} W_{\mathbf{k}\mathbf{k}'} \left\langle c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow} \right\rangle = -\sum_{\mathbf{k}'} W_{\mathbf{k}\mathbf{k}'} \frac{\Delta_{\mathbf{k}'}}{2E_{\mathbf{k}'}} [1 - 2f(E_{\mathbf{k}'})]$$

We used Fermi distribution

$$\left\langle \gamma^{\dagger}_{\mathbf{k}\sigma} \gamma_{\mathbf{k}\sigma} \right\rangle = f(E_{\mathbf{k}}) \,, \qquad f(E) = \frac{1}{e^{E/k_{\mathrm{B}}T} + 1} \,.$$

For our model interaction $\mathbf{k}$ dependence is simple

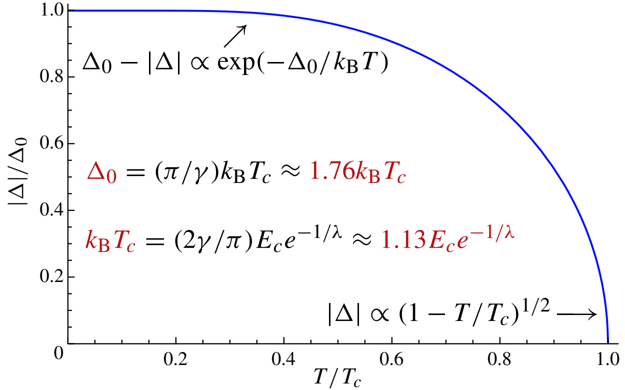
$$W_{\mathbf{k}\mathbf{k}'} = \begin{cases} -W, & \epsilon_{\mathbf{k}} \text{ and } \epsilon_{\mathbf{k}'} < E_c \\ 0, & \text{otherwise} \end{cases} \qquad \Delta_{\mathbf{k}} = \begin{cases} \Delta, & E_{\mathbf{k}} < \sqrt{E_c^2 + |\Delta|^2} \approx E_c \\ 0, & \text{otherwise} \end{cases}$$

With these substitutions the gap equation becomes

$$\Delta = W \Delta \sum_{\mathbf{k}, \epsilon_{\mathbf{k}} < E_c} \frac{1 - 2f(E_{\mathbf{k}})}{2E_{\mathbf{k}}} = \Delta \frac{W}{2} \sum_{\mathbf{k}, \epsilon_{\mathbf{k}} < E_c} \frac{1}{E_{\mathbf{k}}(\Delta)} \tanh \frac{E_{\mathbf{k}}(\Delta)}{2k_{\mathrm{B}}T}$$

It has a trivial solution $\Delta = 0$ corresponding to the normal metal.

THE GAP EQUATION



$$1 = \lambda \int_{|\Delta|}^{E_c} \frac{dE}{\sqrt{E^2 - |\Delta|^2}} \tanh \frac{E}{2k_B T}$$

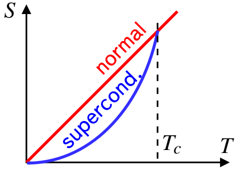
Interaction constant $\lambda = N(0)W$

$\lambda \sim 0.1 - 0.3$ in practical superconductors

HEAT CAPACITY

Only quasiparticles contribute to the entropy

$$S = -k_\mathrm{B} \sum_{\mathbf{k}\sigma} \left[(1 - f(E_\mathbf{k})) \ln(1 - f(E_\mathbf{k})) + f(E_\mathbf{k}) \ln f(E_\mathbf{k}) \right]$$



$$f(E_\mathbf{k}) = \frac{1}{e^{E_\mathbf{k}/k_\mathrm{B}T} + 1}, \qquad E_\mathbf{k} = \sqrt{\xi_\mathbf{k}^2 + |\Delta_\mathbf{k}(T)|^2}$$

The heat capacity

$$C = -T\frac{dS}{dT} = \frac{2}{k_\mathrm{B}} \sum_\mathbf{k} f(E_\mathbf{k})(1 - f(E_\mathbf{k})) \left[\frac{E_\mathbf{k}^2}{T^2} - \frac{1}{2T}\frac{d|\Delta_\mathbf{k}|^2}{dT} \right]$$

For $T \ll T_c$ we have $E \approx \Delta \gg k_\mathrm{B}T$ and $f(E) \approx e^{-E/k_\mathrm{B}T}$.

As a result

$$C = 2\sqrt{2\pi}\, k_\mathrm{B} N(0) \Delta_0 \left(\frac{\Delta_0}{k_\mathrm{B}T} \right)^{3/2} \exp\left(-\frac{\Delta_0}{k_\mathrm{B}T} \right).$$

NON-UNIFORM SUPERCONDUCTORS

Real-space quasiparticle creation and annihilation operators

$$\Psi^\dagger(\mathbf{r},\sigma) = \sum_{\mathbf{k}} e^{-i\mathbf{kr}} c^\dagger_{\mathbf{k}\sigma}, \quad \Psi(\mathbf{r},\sigma) = \sum_{\mathbf{k}} e^{i\mathbf{kr}} c_{\mathbf{k}\sigma}$$

Non-interacting Hamiltonian corresponds to

$$H_0 = \sum_{\mathbf{k}\sigma} \xi_{\mathbf{k}} c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} \quad \rightarrow \quad H_0 = \sum_{\sigma} \int d^3\mathbf{r}\, \Psi^\dagger(\mathbf{r},\sigma) \hat{H}_e \Psi(\mathbf{r},\sigma)$$

where the free particle Hamiltonian

$$\xi_{\mathbf{k}} = \frac{\hbar^2 k^2}{2m} - E_F \quad \rightarrow \quad \hat{H}_e = \frac{1}{2m}\left(-i\hbar\nabla - \frac{e}{c}\mathbf{A}\right)^2 + U(\mathbf{r}) - \mu,$$

which accounts for the magnetic field through the vector potential $\mathbf{A}$ and includes some non-magnetic potential $U(\mathbf{r})$.

BCS THEORY IN COORDINATE SPACE

Non-interacting $\rightarrow$ add interaction $(\Psi^\dagger\Psi^\dagger\Psi\Psi)$ $\rightarrow$ mean-field theory

$\rightarrow$ diagonalize with Bogolubov transformation

$$\Psi(\mathbf{r}\uparrow) = \sum_{\mathbf{k}} \left[u_{\mathbf{k}}(\mathbf{r})\gamma_{\mathbf{k}\uparrow} - v^*_{\mathbf{k}}(\mathbf{r})\gamma^\dagger_{-\mathbf{k}\downarrow} \right]$$

$$\Psi^\dagger(\mathbf{r}\downarrow) = \sum_{\mathbf{k}} \left[u^*_{\mathbf{k}}(\mathbf{r})\gamma^\dagger_{-\mathbf{k}\downarrow} + v_{\mathbf{k}}(\mathbf{r})\gamma_{\mathbf{k}\uparrow} \right]$$

Here **k** can be any enumeration of states, not necessarily wave vector.

[For uniform case $u_{\mathbf{k}}(\mathbf{r}) = u_{\mathbf{k}}e^{i\mathbf{kr}}$ and $v_{\mathbf{k}}(\mathbf{r}) = v_{\mathbf{k}}e^{i\mathbf{kr}}$]

$+$ completeness/orthogonality conditions on $u_{\mathbf{k}}(\mathbf{r})$ and $v_{\mathbf{k}}(\mathbf{r})$ from Fermi commutation relations for Ψ and γ.

BOGOLUBOV – DE GENNES EQUATIONS

Diagonalization condition

$$\begin{cases} \hat{H}_e\, u_{\mathbf{k}}(\mathbf{r}) + \Delta(\mathbf{r})\, v_{\mathbf{k}}(\mathbf{r}) = E_{\mathbf{k}}\, u_{\mathbf{k}}(\mathbf{r}) \\ \Delta^*(\mathbf{r})\, u_{\mathbf{k}}(\mathbf{r}) - \hat{H}_e^*\, v_{\mathbf{k}}(\mathbf{r}) = E_{\mathbf{k}}\, v_{\mathbf{k}}(\mathbf{r}) \end{cases}$$

$$\hat{H}_e = \frac{1}{2m}\left(-i\hbar\nabla - \frac{e}{c}\mathbf{A}\right)^2 + U(\mathbf{r}) - \mu$$

Self-consistency equation

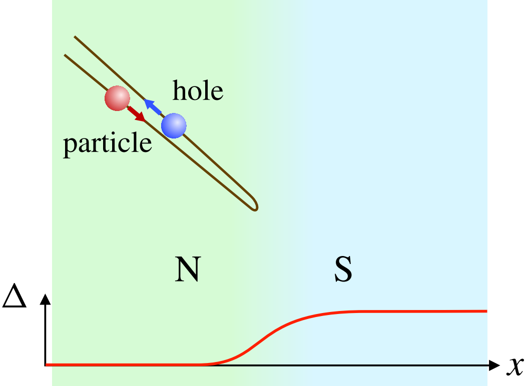
$$\Delta(\mathbf{r}) = W \sum_{\mathbf{k},\epsilon_{\mathbf{k}}<E_c} u_{\mathbf{k}}(\mathbf{r}) v_{\mathbf{k}}^*(\mathbf{r})[1 - 2f(E_{\mathbf{k}})]$$

+ completeness/orthogonality conditions on $u_{\mathbf{k}}(\mathbf{r})$ and $v_{\mathbf{k}}(\mathbf{r})$

+ Maxwell equations to connect current and field

Andreev reflection

Alexander Andreev

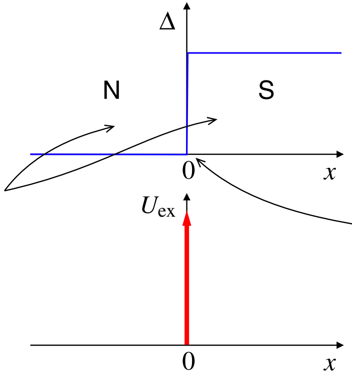


MODEL OF NS(NIS) INTERFACE

Bogolubov – de Gennes equations

$$-\frac{\hbar^2}{2m}\left(\nabla - \frac{ie}{\hbar c}\mathbf{A}\right)^2 u(\mathbf{r}) + [U_{\mathrm{ex}}(\mathbf{r}) - E_F]\, u(\mathbf{r}) + \Delta(\mathbf{r}) v(\mathbf{r}) = \epsilon u(\mathbf{r}),$$

$$\frac{\hbar^2}{2m}\left(\nabla + \frac{ie}{\hbar c}\mathbf{A}\right)^2 v(\mathbf{r}) - [U_{\mathrm{ex}}(\mathbf{r}) - E_F]\, v(\mathbf{r}) + \Delta^*(\mathbf{r}) u(\mathbf{r}) = \epsilon v(\mathbf{r}).$$



Use uniform bulk solutions on left and right

Join $(u,\, v,\, \partial_x u,\, \partial_x v)$ at the interface

$$U_{\mathrm{ex}}(\mathbf{r}) = I\delta(x)$$

INCIDENT AND REFLECTED QUASIPARTICLES

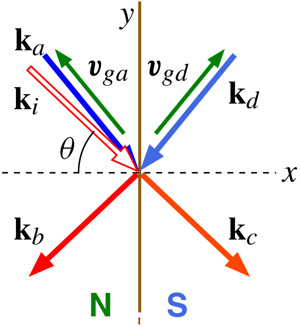
Incident particle excitation with energy $\epsilon > |\Delta|$

Amplitudes (for $I = 0$)

$a = A^- / A^+, \quad b = 0$

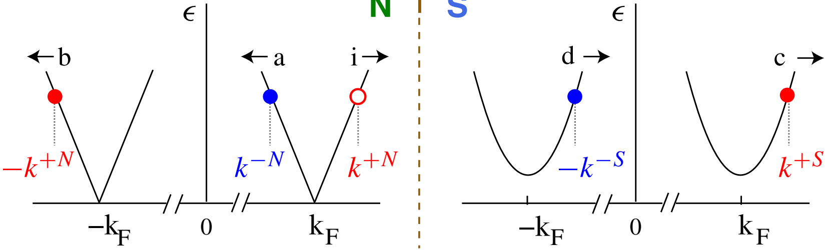
$c = 1/A^+, \quad d = 0$

$$A^{\pm} = \frac{1}{\sqrt{2}} \left(1 \pm \frac{\sqrt{\epsilon^2 - |\Delta|^2}}{\epsilon} \right)^{1/2}$$



$$k^{\pm N} = k_F \pm \frac{\epsilon}{\hbar v_F}$$

$$k^{\pm S} = k_F \pm \frac{\sqrt{\epsilon^2 - |\Delta|^2}}{\hbar v_F}$$

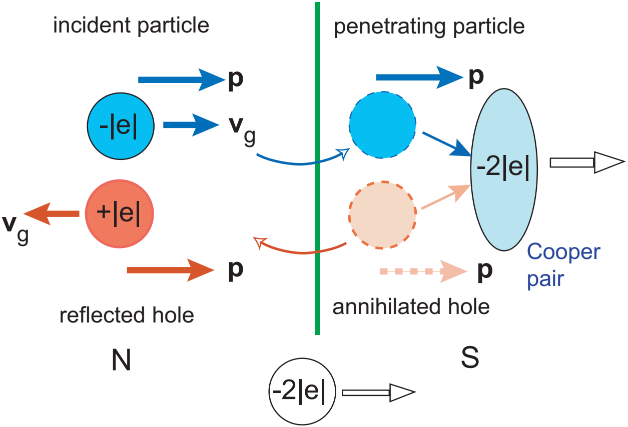


SUBGAP STATES

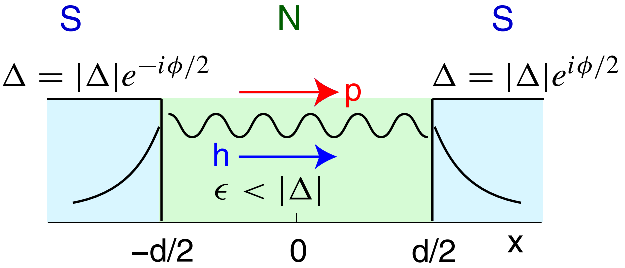
For an excitation coming from the normal side with $\epsilon < |\Delta|$

Andreev reflection probability

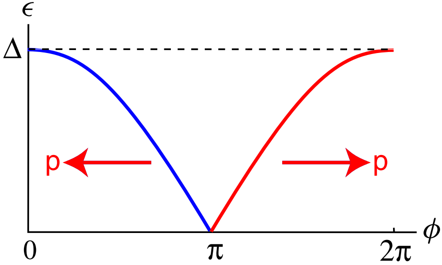
$$|a|^2 = |A^-/A^+|^2 = 1 \qquad A^{\pm} = \frac{1}{\sqrt{2}}\left(1 \pm i\frac{\sqrt{|\Delta|^2 - \epsilon^2}}{\epsilon}\right)^{1/2}$$



ANDREEV BOUND STATES



Spectrum for short junction ($d < \xi$)



$$\epsilon = \mp |\Delta| \cos \frac{\phi}{2}$$

NEGATIVE ENERGIES

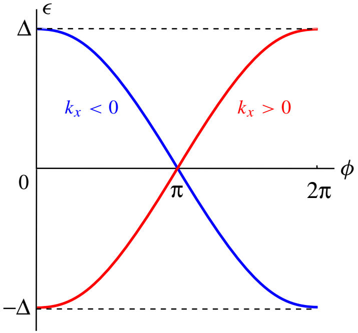
Bogolubov – de Gennes equations possess an important property: If

$$\begin{pmatrix} u(\mathbf{r}) \\ v(\mathbf{r}) \end{pmatrix} \quad \text{is a solution for the energy} \quad \epsilon$$

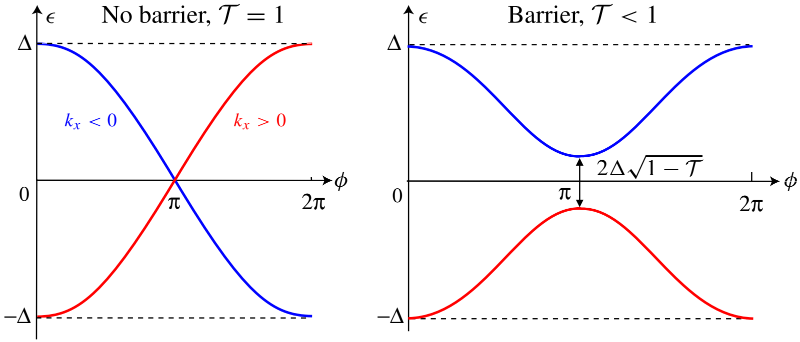
then

$$\begin{pmatrix} v(\mathbf{r})^* \\ -u(\mathbf{r})^* \end{pmatrix} \quad \text{is a solution for the energy} \quad -\epsilon$$

Thus formally we can introduce negative energies and "Dirac sea" of excitations.



SUPERCONDUCTOR–INSULATOR–SUPERCONDUCTOR (SIS) CONTACT



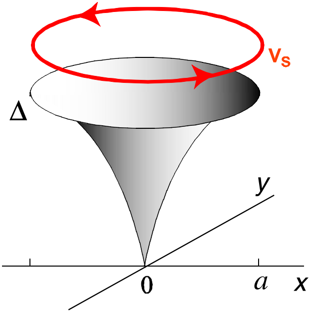
$$\epsilon = \pm\epsilon_\phi = \pm|\Delta|\sqrt{1 - \mathcal{T}\sin^2(\phi/2)}$$

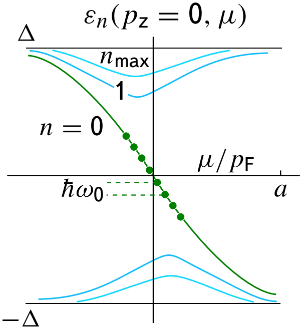
Conduction channel transmission coefficient $\mathcal{T}$

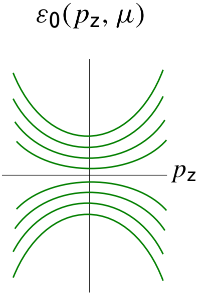
$$\frac{1}{R_N} = \frac{\mathcal{T}}{R_0}, \quad \text{quantum of resistance} \quad R_0 = \frac{\pi\hbar}{e^2} \approx 12.9\ \mathrm{k\Omega}$$

BOUND FERMION STATES IN THE VORTEX CORE

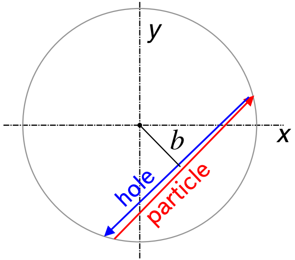
Caroli, de Gennes, Matricon 1964







Andreev reflection



Radial quantum number n $(n_{max} \sim a/\xi)$.

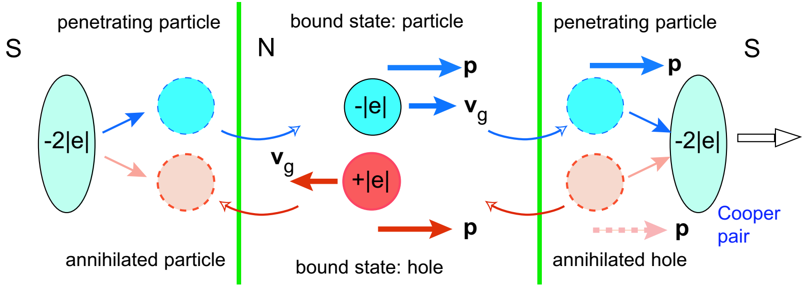
Angular momentum $\mu = b\, p_\perp$, quantized.

$$\mu/\hbar = \begin{cases} m + 1/2, & \text{s-wave superconductors} \\ m, & \text{superfluid } ^3\text{He} \end{cases}$$

Minigap $\omega_0 \sim \dfrac{\Delta}{a\, p_F} \sim \dfrac{1}{\hbar} \dfrac{\Delta^2}{E_F} \ll \dfrac{\Delta}{\hbar}$.

Anomalous (crossing zero) branch $n = 0$.

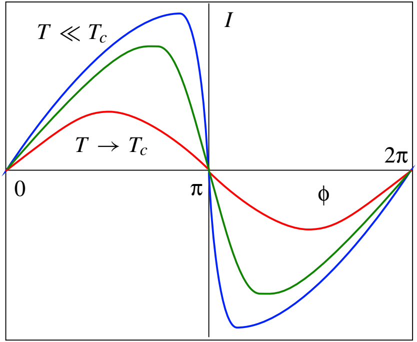
SUPERCURRENT VIA ANDREEV BOUND STATES



SUPERCURRENT IN THE POINT CONTACT

For the contact with resistance R_N in the normal state supercurrent is

$$I_s = \frac{\pi|\Delta|\sin(\phi/2)}{eR_N} \tanh \frac{|\Delta|\cos(\phi/2)}{2k_B T}$$



$T \ll T_c$: $\quad I_s \approx I_c \sin(\phi/2)\,\mathrm{sign}\cos(\phi/2)$

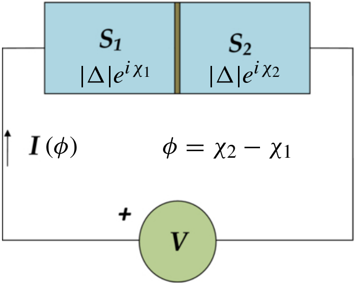
$$I_c = \frac{\pi|\Delta|}{eR_N}$$

$T \to T_c$: $\quad I_s \approx I_c \sin(\phi)$

$$I_c = \frac{\pi|\Delta|^2}{4k_B T e R_N}$$

Josephson effect and weak links

Brian Josephson



DC Josephson effect

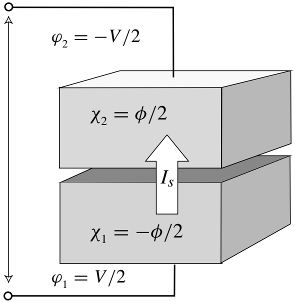
$$I_s = I_c \sin\phi$$

I_c is the critical Josephson current

AC Josephson effect

$$\hbar \frac{d\phi}{dt} = 2eV$$

WEAKLY COUPLED SUPERCONDUCTORS (Feynman model)



Wave functions of the Cooper-pair condensate are uniform

$$\psi_1 = N_1^{1/2} e^{i\chi_1}, \quad \psi_2 = N_2^{1/2} e^{i\chi_2}$$

N_1, N_2 – number of Cooper pairs.

When no coupling $\psi_\alpha = \text{const}\,(t)$

$$i\hbar \frac{\partial \psi_\alpha}{\partial t} = E_\alpha \psi_\alpha = 0 \;\Rightarrow\; E_\alpha = 0, \quad \alpha = 1, 2$$

With coupling $-K$ between the superconductors

$$i\hbar \frac{\partial \psi_1}{\partial t} = (E_1 + e^* V/2)\psi_1 - K\psi_2$$
$$i\hbar \frac{\partial \psi_2}{\partial t} = (E_2 - e^* V/2)\psi_2 - K\psi_1$$

Here $e^* = 2e$ is the charge of the Cooper pair.

DC JOSEPHSON EFFECT

We obtain

$$\hbar \frac{dN_1}{dt} = -2K\sqrt{N_1 N_2}\sin(\chi_2 - \chi_1)$$
$$\hbar \frac{dN_2}{dt} = 2K\sqrt{N_1 N_2}\sin(\chi_2 - \chi_1)$$

This gives the charge conservation $N_1 + N_2 =$ const together with the relation

$$I_s = I_c \sin\phi$$

where

$$I_s = e^* \frac{dN_2}{dt} = -e^* \frac{dN_1}{dt}$$

is the current flowing from the first into the second electrode,

$$I_c = 4eK\sqrt{N_1 N_2}/\hbar$$

is the critical Josephson current, while $\phi = \chi_2 - \chi_1$ is the phase difference.

AC JOSEPHSON EFFECT

For phases we find

$$\hbar N_2 \frac{d\chi_2}{dt} = eVN_2 + K\sqrt{N_1 N_2}\cos(\chi_2 - \chi_1)$$
$$\hbar N_1 \frac{d\chi_1}{dt} = -eVN_1 + K\sqrt{N_1 N_2}\cos(\chi_2 - \chi_1)$$

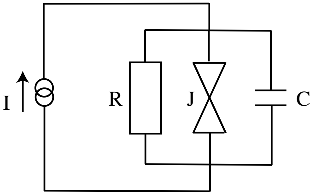
Subtracting we get

$$\hbar(N_2 - N_1)\frac{d(\chi_2 - \chi_1)}{dt} = 2eV(N_2 - N_1)$$

or

$$\hbar \frac{d\phi}{dt} = 2eV$$

CAPACITIVELY AND RESISTIVELY SHUNTED JUNCTION



$$I = C\frac{\partial V}{\partial t} + \frac{V}{R} + I_c \sin\phi, \qquad V = \frac{\hbar}{2e}\frac{\partial \phi}{\partial t}$$

$$I = \frac{\hbar C}{2e}\frac{\partial^2 \phi}{\partial t^2} + \frac{\hbar}{2eR}\frac{\partial \phi}{\partial t} + I_c \sin\phi$$

Mechanical analogue:

$$M\frac{\partial^2 \phi}{\partial t^2} = -\eta\frac{\partial \phi}{\partial t} - \frac{\partial U(\phi)}{\partial \phi}$$

Particle with coordinate ϕ and with the "mass" $M = \frac{\hbar^2 C}{4e^2} = \frac{\hbar^2}{8E_C}$

Medium with a viscosity $\eta = \frac{\hbar^2}{4e^2 R} = \frac{\hbar^2}{8E_C RC}$,

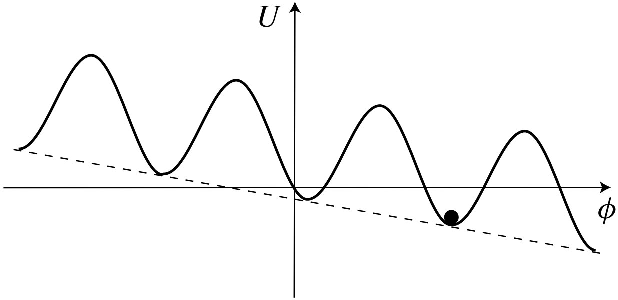
$$\boxed{E_C = \frac{e^2}{2C}}$$

Potential $U(\phi) = E_J(1 - \cos\phi) - (\hbar I/2e)\phi = E_J\,(1 - \cos\phi - \phi I/I_c)$

$$\boxed{E_J = \frac{\hbar I_c}{2e}}$$

WASHBOARD POTENTIAL

$$U(\phi) = E_J\,(1 - \cos\phi - \phi I/I_c)\,, \qquad E_J = \frac{\hbar I_c}{2e}$$



Small oscillations around the minimum:

$$1 - \cos\phi = 2\sin^2(\phi/2) \approx \phi^2/2, \quad U(\phi) \approx \frac{E_J\phi^2}{2}$$

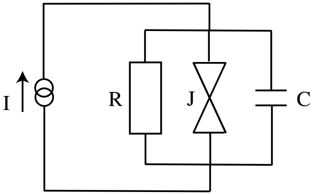
Plasma frequency

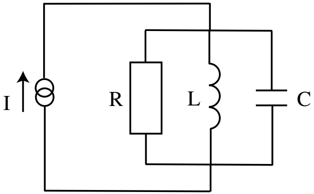
$$\omega_p = \sqrt{\frac{E_J}{M}} = \frac{\sqrt{8E_JE_C}}{\hbar}$$

Quality factor

$$Q = \omega_p RC$$

EFFECTIVE INDUCTANCE





$$|\phi| \ll 2\pi \;\Rightarrow\; I \approx I_c\,\phi$$

$$V = \frac{\hbar}{2e}\frac{\partial\phi}{\partial t} \approx \frac{\hbar}{2eI_c}\frac{\partial I_J}{\partial t} \qquad\qquad V = \frac{1}{c}\frac{\partial\Phi}{\partial t} = \frac{L}{c^2}\frac{\partial I_L}{\partial t}$$

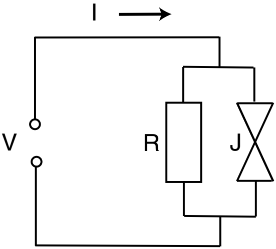
The effective kinetic inductance of the Josephson junction

$$L_J = \frac{\hbar c^2}{2eI_c}$$

Critical current I_c depends e.g. on temperature $\Rightarrow$ detectors.

Nonlinear inductance at large ϕ.

VOLTAGE BIAS



Let us consider the case when the constant voltage V is applied to the junction. In this case

$$\frac{\partial\phi}{\partial t} = \frac{2e}{\hbar}V = \omega_J = \text{const}, \qquad \phi = \phi_0 + \omega_J t$$

Total current

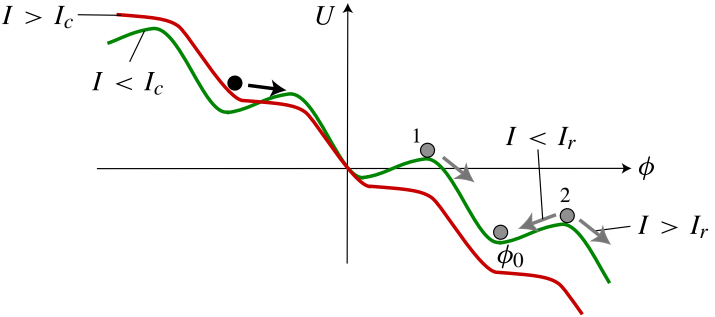
$$I = \frac{V}{R} + I_c \sin(\phi_0 + \omega_J t)$$

and the average current has simple ohmic behavior for any damping

$$\bar{I} = V/R\,.$$

Thus for observation of non-trivial dynamics of Josephson junctions the current bias is essential.

DYNAMICS OF JOSEPHSON JUNCTIONS



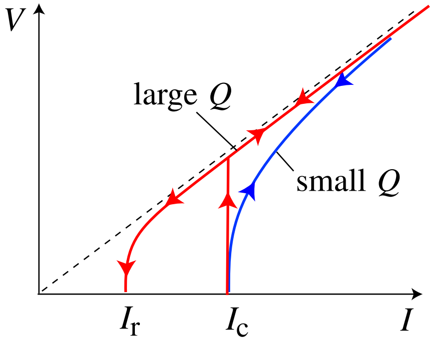
Large damping $$\omega_p^{-2}\frac{\partial^2\phi}{\partial t^2} + Q^{-1}\omega_p^{-1}\frac{\partial\phi}{\partial t} + \sin\phi = \frac{I}{I_c}$$

$$Q^{-1}\omega_p^{-1} \gg \omega_p^{-2}\omega_J \;\Rightarrow\; Q = RC\omega_p \ll \omega_p/\omega_J \;\Rightarrow\; RC \ll \omega_J^{-1}$$

$\Rightarrow$ no inertia (capacitance)

Small damping $\Rightarrow$ hysteretic behavior

CURRENT–VOLTAGE RELATIONS



Small Q (no C) – exact solution:

$$\bar{V} = R\sqrt{I^2 - I_c^2}$$

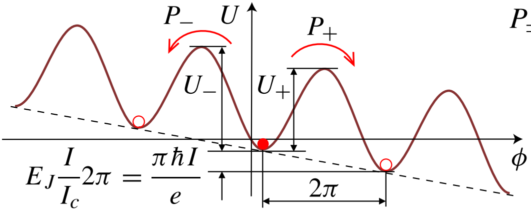
Large Q: rapid slide down with almost constant voltage $V \approx \bar{V}$.
Expanding for small oscillations of phase

$$\phi = \omega_J t + \delta\phi(t), \quad \omega_J = 2e\bar{V}/\hbar, \quad \delta\phi \ll 1$$

leads to

$$\bar{V} = IR\,, \qquad I_r \sim I_c/Q$$

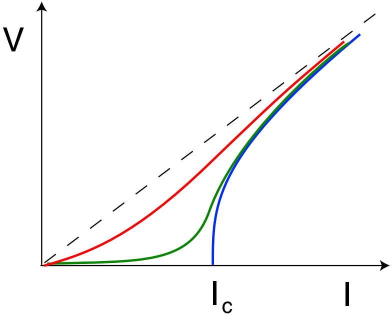
THERMAL FLUCTUATIONS: OVERDAMPED JUNCTION



$$P_\pm = \omega_a \exp\left[-\frac{U_\pm}{k_\mathrm{B} T}\right]$$

$$= \omega_a \exp\left[-\frac{U_0 \mp (\pi \hbar I / 2e)}{k_\mathrm{B} T}\right]$$

$$U_0 \approx 2E_J \;\; \text{for} \;\; I \ll I_c$$

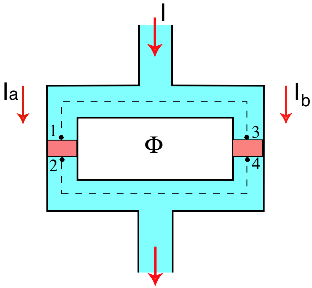


$$\bar{V} = \frac{\hbar}{2e} \overline{\partial \phi / \partial t} = \frac{\hbar}{2e} 2\pi (P_+ - P_-)$$

$$= I R \frac{E_J}{k_\mathrm{B} T} \exp\left(-\frac{2E_J}{k_\mathrm{B} T}\right)$$

SUPERCONDUCTING QUANTUM INTERFERENCE DEVICES

dc SQUID:



$$\mathbf{j}_s = -\frac{e^2 n_s}{mc}\left(\mathbf{A} - \frac{\hbar c}{2e}\nabla\chi\right) = 0$$

$$\chi_3 - \chi_1 + \chi_2 - \chi_4 - \frac{2e}{\hbar c}\left(\int_1^3 \mathbf{A}\cdot d\mathbf{l} + \int_4^2 \mathbf{A}\cdot d\mathbf{l}\right) = 0$$

$$\underbrace{(\chi_2 - \chi_1)}_{\phi_a} - \underbrace{(\chi_4 - \chi_3)}_{\phi_b} = \frac{2e}{\hbar c}\oint_{1342}\mathbf{A}\cdot d\mathbf{l} = \frac{2\pi\Phi}{\Phi_0}$$

Total current

$$I = I_a + I_b = I_c\sin\phi_a + I_c\sin\phi_b = 2I_c\cos\left(\frac{\pi\Phi}{\Phi_0}\right)\sin\left(\phi_a - \frac{\pi\Phi}{\Phi_0}\right)$$

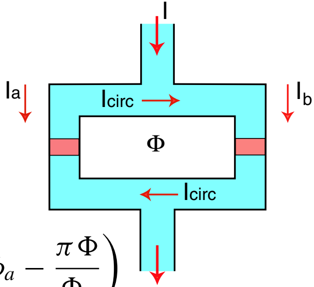
The maximum current depends on the magnetic flux through the loop

$$I_{c,\mathrm{SQUID}} = 2\left|I_c\cos\left(\frac{\pi\Phi}{\Phi_0}\right)\right|$$

INFLUENCE OF THE SQUID INDUCTANCE

Extra flux from the circulating current

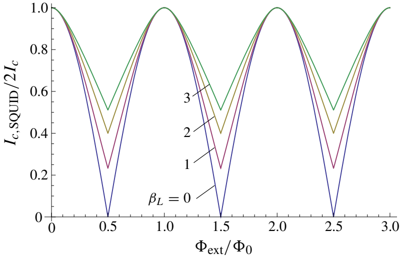
$$I_{\mathrm{circ}} = \frac{I_b - I_a}{2} = \frac{I_c}{2}\left[\sin\left(\phi_a - \frac{2\pi\Phi}{\Phi_0}\right) - \sin\phi_a\right]$$



in the SQUID loop

$$\Phi = \Phi_{\mathrm{ext}} + \frac{L I_{\mathrm{circ}}}{c} = \Phi_{\mathrm{ext}} - \beta_L \frac{\Phi_0}{2\pi}\sin\left(\frac{\pi\Phi}{\Phi_0}\right)\cos\left(\phi_a - \frac{\pi\Phi}{\Phi_0}\right)$$

Dimensionless parameter $\beta_L = \frac{2eLI_c}{\hbar c^2} = \frac{L}{L_J} \gtrsim 1$ in practice



Together with

$$I = 2I_c\cos\left(\frac{\pi\Phi}{\Phi_0}\right)\sin\left(\phi_a - \frac{\pi\Phi}{\Phi_0}\right)$$

determines maximum current

SHAPIRO STEPS

Microwave irradiation of the junction $V = V_0 + V_1 \cos(\omega t)$ and

$$\phi = \frac{2e}{\hbar} \int_0^t V(t')\, dt' = \phi_0 + \omega_J t + a \sin(\omega t), \quad \omega_J = \frac{2e}{\hbar} V_0, \quad a = \frac{2e}{\hbar} \frac{V_1}{\omega}$$

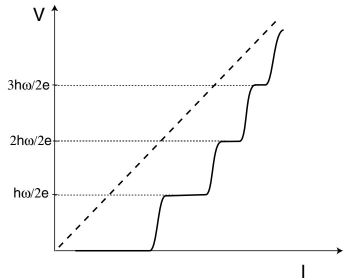
The supercurrent

$$I = I_c \sin\phi = I_c \sum_{n=-\infty}^{\infty} (-1)^n J_n(2eV_1/\hbar\omega) \sin(\phi_0 + \omega_J t - n\omega t)$$

When

$$\omega_J = n\omega, \quad \text{i.e.} \quad V_0 = n(\hbar\omega/2e)$$

the supercurrent has a dc component $I_n = I_c J_n(2eV_1/\hbar\omega) \sin(\phi_0 + \pi n)$.



QUANTUM DYNAMICS OF JOSEPHSON JUNCTIONS

Analogy

Josephson junction ↔ particle in the washboard potential

can be extended to quantum-mechanical description.

If ϕ is the coordinate of the a particle, then the momentum operator is

$$\hat{p}_\phi = -i\hbar \frac{\partial}{\partial \phi}$$

The Shrödinger equation for the wave function Ψ is

$$\hat{\mathcal{H}}\Psi = \left[\frac{\hat{p}_\phi^2}{2M} + U(\phi)\right]\Psi = E\Psi$$

with

$$M = \frac{\hbar^2 C}{4e^2} = \frac{\hbar^2}{8E_C}, \quad U(\phi) = E_J(1 - \cos\phi) - (\hbar I/2e)\phi$$

Thus the Hamiltonian is

$$\hat{\mathcal{H}} = -4E_C \frac{\partial^2}{\partial \phi^2} + E_J\left(1 - \cos\phi - \phi I/I_c\right)$$

REQUIREMENTS FOR JUNCTION PARAMETERS

Kinetic energy $\leftrightarrow$ The charging energy of the capacitor

The operator of charge $\hat{Q}$ on the capacitor:

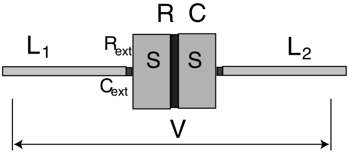
$$\frac{\hat{p}_\phi^2}{2M} = \frac{\hat{Q}^2}{2C}, \quad \hat{Q} = -2ie\frac{\partial}{\partial\phi}, \quad [\hat{Q},\phi]_- = \frac{2e}{\hbar}[\hat{p}_\phi,\phi]_- = -2ie$$

Quantum uncertainty in phase $\Delta\phi$ and in charge ΔQ: $\Delta\phi\Delta Q \sim 2e$.

Effects for $Q \sim e$ are important:

$$E_C = \frac{e^2}{2C} \gg k_B T, \quad C = \frac{\epsilon A}{4\pi d} \sim \frac{10\,(100\,\text{nm})^2}{4\pi \cdot 1\,\text{nm}} \sim 10^{-15}\,\text{F} \quad \Rightarrow \quad T \ll 1\,K$$

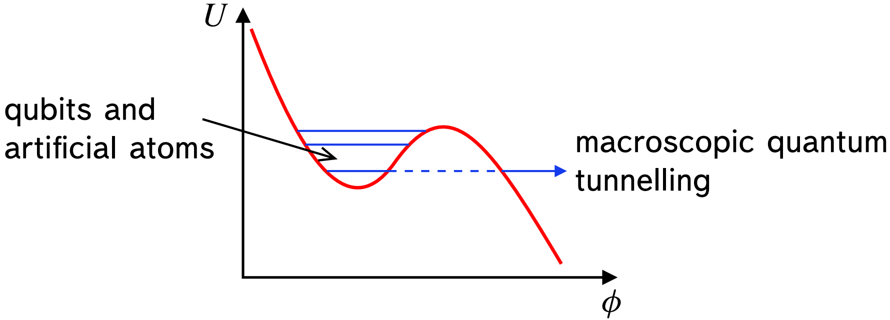
$$E_C \gg \frac{\hbar}{\Delta t} = \frac{\hbar}{RC} \quad \Rightarrow \quad R \gg \frac{2\hbar}{e^2} \sim R_0 = \frac{\pi\hbar}{e^2}$$



$$C_{ext} \ll C, \quad R_0 \ll R_{ext} \ll R$$

SUPERCONDUCTING QUANTUM ELECTRONICS

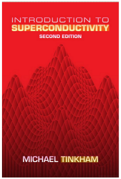
Rapidly developing field. One of key players in quantum engineering and quantum information processing.



CONCLUSIONS

- Superconductivity is ubiquitous in conducting systems: metals, alloys, dirty and disordered systems, organic materials, 2D system...
- Superconductivity originates in the Cooper pairing of conduction carriers. In many systems the attractive interaction is mediated by phonons and pairing occurs in spin-singlet, orbital momentum zero state. But more and more unconventional systems are being discovered.
- Besides zero resistivity, superconductors possess broad range of interesting and practically important properties: magnetic, thermal, etc.
- Nanotechnology opened a new world in studies and applications of superconductors, in particular due to good matching to important physical length scales. In nanodevices Josephson effect and Andreev bounds states usually play key roles.
- Superconductors provide access to quantum-mechanical coherence at macroscopic length scales: a base for revolutionizing the world with quantum technologies.

LITERATURE



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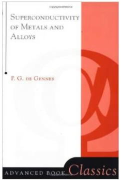

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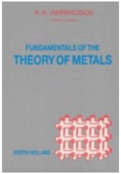

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